Chapter 37 Diffraction Patterns

Physics II – Part III

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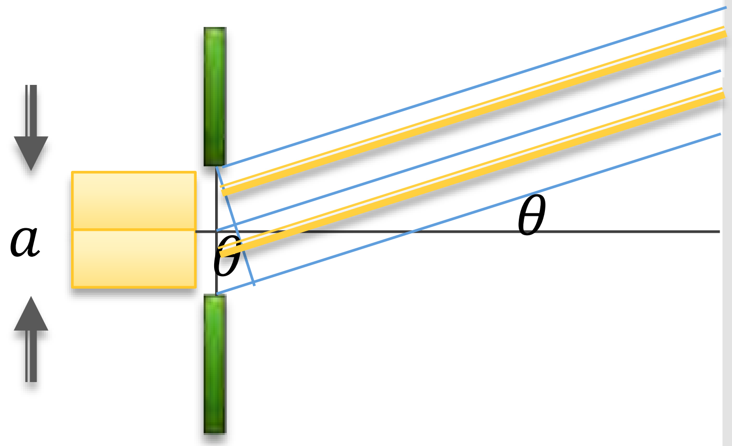
Diffraction Pattern of Single Slit

Single Slit Diffraction

Simple concept about the bright or the dark area



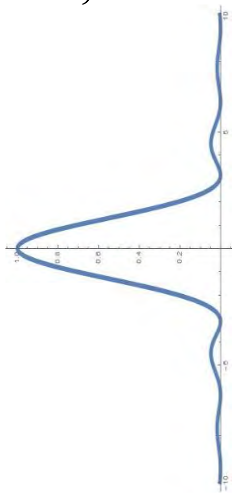
Dark area: $\frac{a}{2}\sin(\theta) = \frac{\lambda}{2}$

Bright area: $\frac{a}{2}\sin(\theta) = 3\lambda/4$

The central area is bright since $\Delta\phi = 0$

The condition gives you the size of the central bright area, calculated by the positions of the two first-dark areas.

$$\frac{a}{2}\sin(\theta) = \frac{\lambda}{2} \& \sin(\theta) \cong \tan(\theta) \rightarrow \frac{y}{L} \cong \frac{\lambda}{a}$$



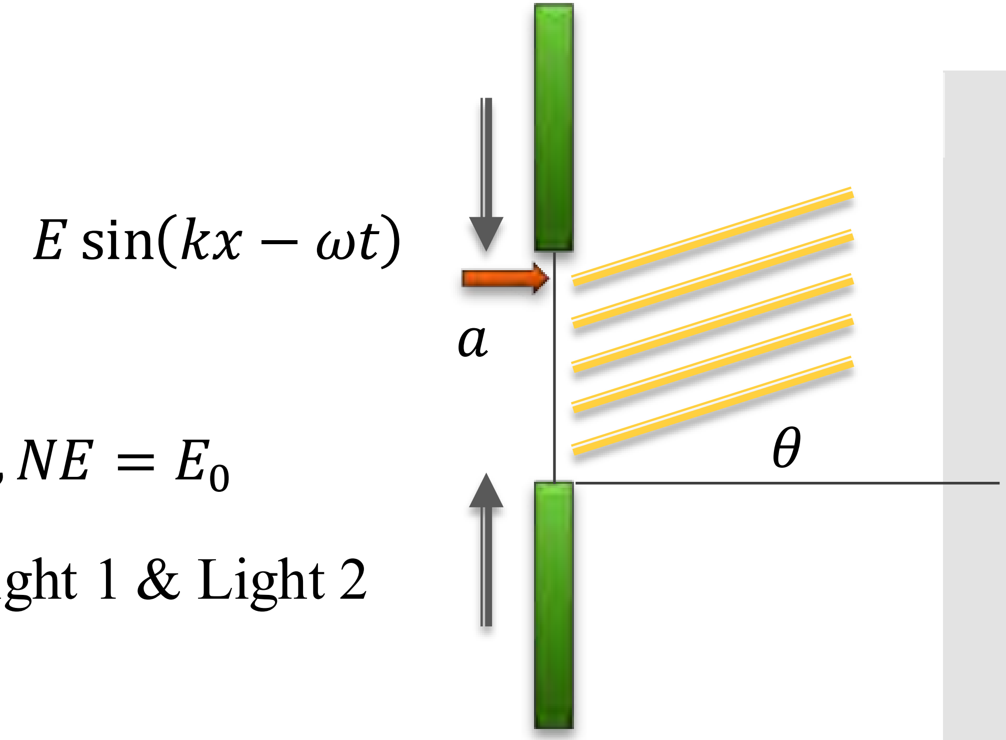
The size of the central bright area is $2y = 2L\frac{\lambda}{a}$.

Ref:

Single Slit Diffraction

Intensity of The Diffraction Pattern of Single Slit

The initial light $E_0 \sin(kx - \omega t)$ is divided to N waves of $E \sin(kx - \omega t)$.



$$I_{total} = \varepsilon_0 C \langle E_0^2 \sin^2(kx - \omega t) \rangle, NE = E_0$$

The phase difference between Light 1 & Light 2

$$\delta = 2\pi \frac{\frac{a}{N} \sin(\theta)}{\lambda}$$

The phase difference between Light 1 & Light N

$$\phi = (N-1)\delta = 2\pi \frac{N-1}{N} \frac{a \sin(\theta)}{\lambda} \cong N\delta = 2\pi \frac{a \sin(\theta)}{\lambda}$$

Ref:

Intensity of The Diffraction Pattern of Single Slit

Single Slit Diffraction

Superposition of the divided light waves:

$$f_{total}(x,t) = E\sin(kx-\omega t) + E\sin(kx-\omega t+\delta) + \cdots$$

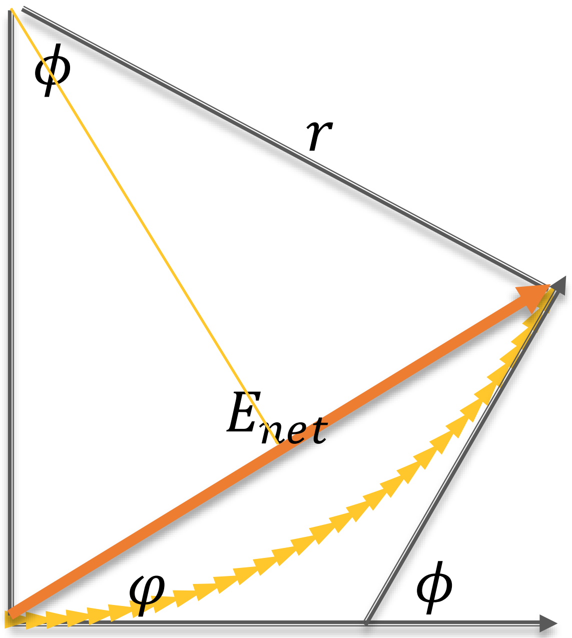
$$+E\sin(kx-\omega t+(N-1)\delta) = E_{net}\sin(kx-\omega t+\varphi)$$

Use phasor to find out the total wave function. It is that to determine E_{net} and φ.

$$NE = E_0 = r\phi \rightarrow r = \frac{NE}{\phi} = \frac{E_0}{\phi}$$

$$E_{net} = 2r\sin\left(\frac{\phi}{2}\right) = 2\frac{E_0}{\phi}\sin\left(\frac{\phi}{2}\right)$$

$$\varphi = \frac{\phi}{2}$$



Ref:

Intensity of The Diffraction Pattern of Single Slit

Single Slit Diffraction

Superposition of the divided light waves:

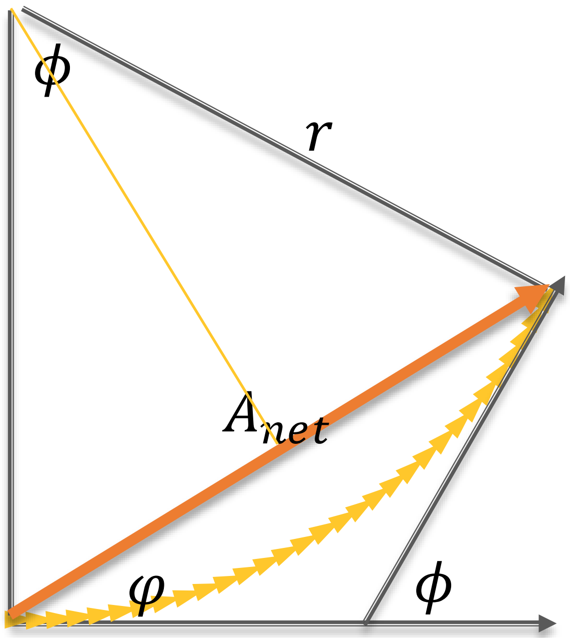
$$f_{total}(x,t) = 2\frac{E_0}{\phi}\sin\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

$$I = \varepsilon_0 C\left\langle\left(\frac{E_0}{\frac{\phi}{2}}\right)^2 \sin^2\left(\frac{\phi}{2}\right)\sin^2(kx - \omega t)\right\rangle$$

$$I = I_0\frac{\sin^2\left(\frac{\phi}{2}\right)}{\left(\frac{\phi}{2}\right)^2}$$

$$\phi = 2\pi\frac{a\sin(\theta)}{\lambda}$$

$$I = I_0\frac{\sin^2\left(\frac{\pi a\sin(\theta)}{\lambda}\right)}{\left(\frac{\pi a\sin(\theta)}{\lambda}\right)^2}$$



Ref:

Intensity of Two Slit Diffraction Pattern

Real Two Slit Diffraction Pattern

Interference of light from the two slits:

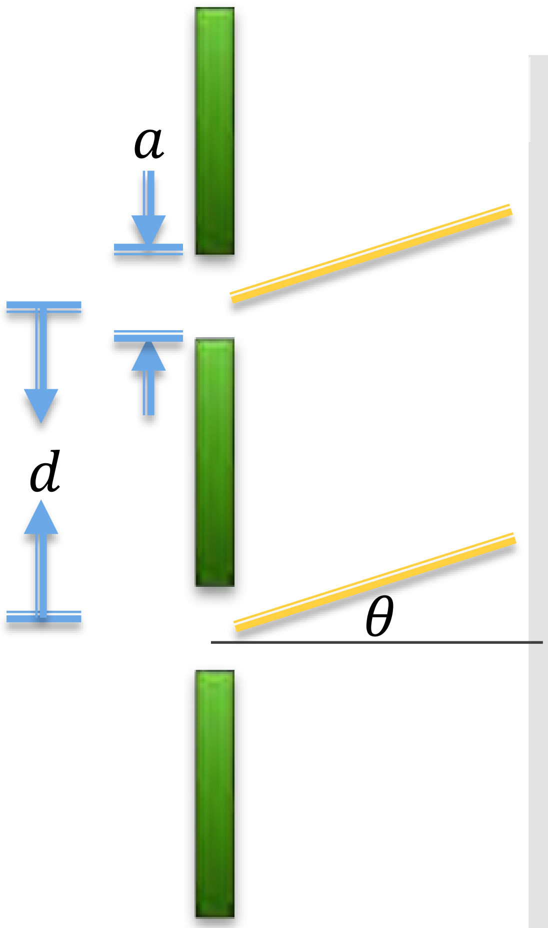
$$\varphi_I = 2\pi \frac{d \sin(\theta)}{\lambda}$$

$$f_t = \frac{E_0}{2} \sin(kx - \omega t) + \frac{E_0}{2} \sin(kx - \omega t + \varphi_I)$$

$$f_t = E_0 \sin\left(kx - \omega t + \frac{\varphi_I}{2}\right) \cos\left(\frac{\varphi_I}{2}\right)$$

$$I_I = \varepsilon_0 C \cos^2\left(\frac{\varphi_I}{2}\right) \left\langle E_0^2 \sin^2\left(kx - \omega t + \frac{\varphi_I}{2}\right)\right\rangle$$

$$I_I = I_0 \cos^2\left(\frac{\varphi_I}{2}\right) = I_0 \cos^2\left(\frac{d\,\pi \sin(\theta)}{\lambda}\right)$$



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Real Two Slit Diffraction Pattern

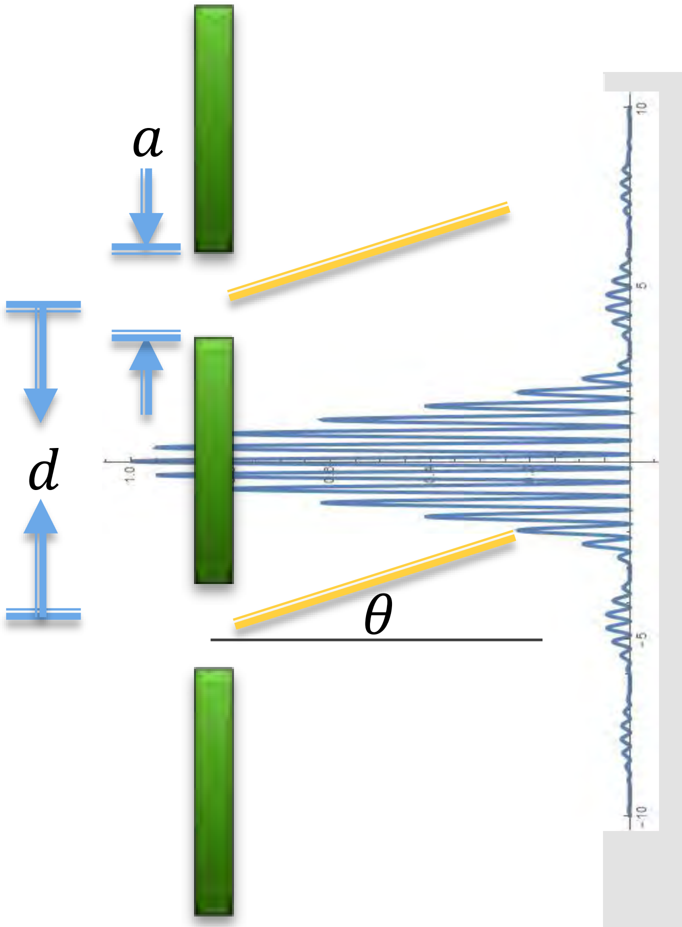
Intensity of Two Slit Diffraction Pattern

$$I_I = I_0 \cos^2\left(\frac{d\,\pi\sin(\theta)}{\lambda}\right)$$

Considering the single slit diffraction, since $d > a$, the width of the bright area of the two-slit interference shall be smaller than that of the single slit diffraction.

$$I = I_0 \cos^2\left(\frac{d\,\pi\sin(\theta)}{\lambda}\right)\frac{\sin^2\left(\frac{\pi a\sin(\theta)}{\lambda}\right)}{\left(\frac{\pi a\sin(\theta)}{\lambda}\right)^2}$$

$$I = I_0 \cos^2\left(\frac{d\pi y}{\lambda L}\right)\frac{\sin^2\left(\frac{\pi a y}{\lambda L}\right)}{\left(\frac{\pi a y}{\lambda L}\right)^2}$$



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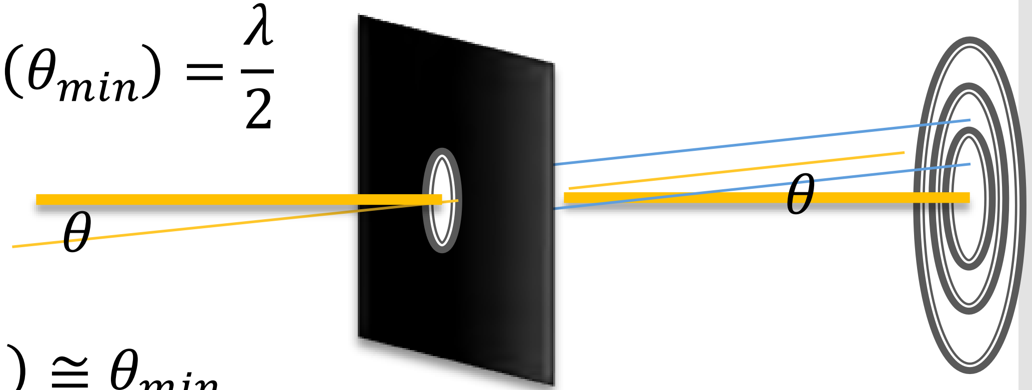
Resolution of Single Slit

The dark ring exists at the condition of

$$I = I_0 \frac{\sin^2\left(\frac{\pi a \sin(\theta)}{\lambda}\right)}{\left(\frac{\pi a \sin(\theta)}{\lambda}\right)^2} \qquad \frac{a}{2}\sin(\theta_{min}) = \frac{\lambda}{2}$$

$$a \sin(\theta_{min}) = 1.22\lambda$$



Because $\theta_{min} \ll 1$, $\sin(\theta_{min}) \cong \theta_{min}$

$$\theta_{min} \cong \frac{1.22\lambda}{a}$$

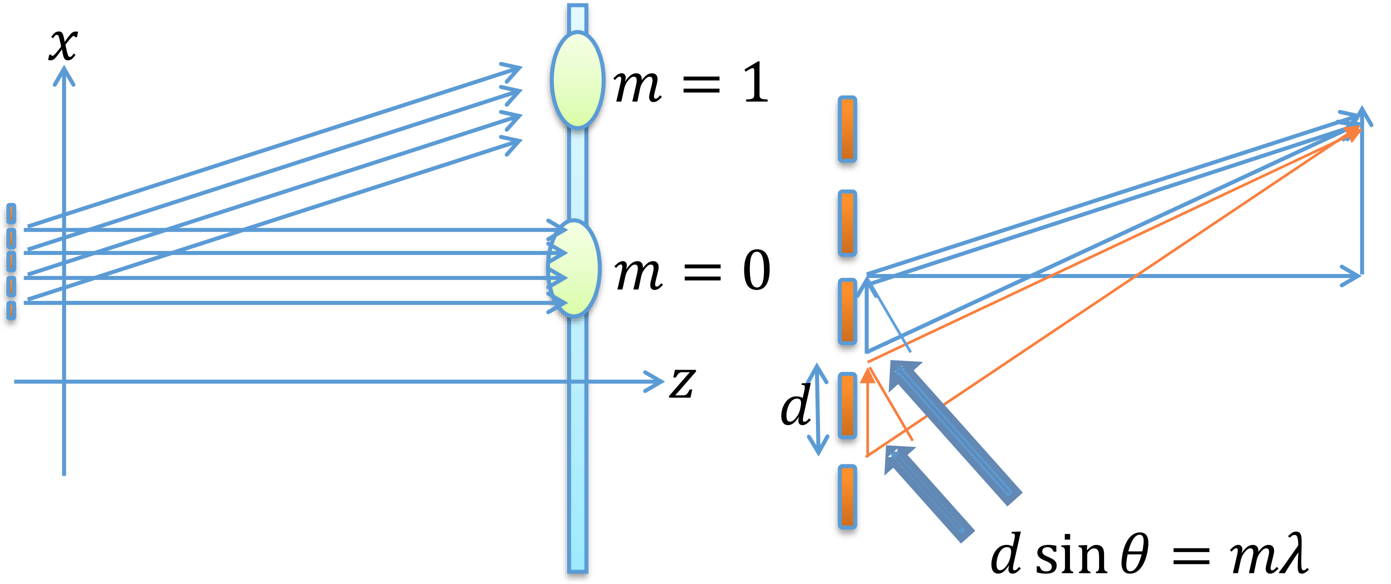
Sometimes we use D rather than a to represent the diameter of a viewport such as a telescope.

$$\theta_{min} \cong \frac{1.22\lambda}{D}$$

Ref:

Diffraction Caused by Grating

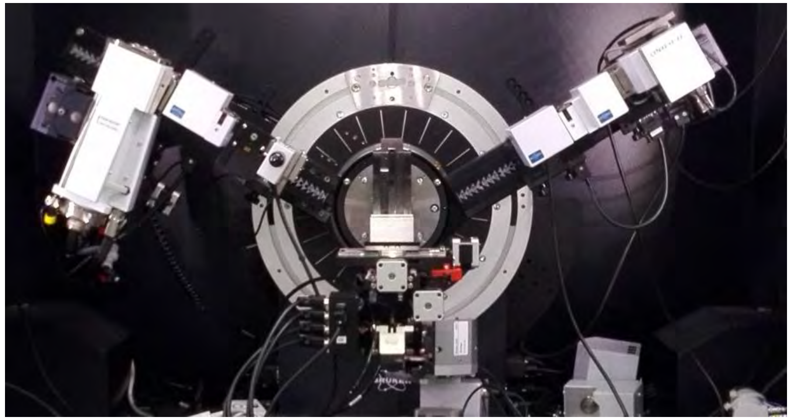
The Grating



Here we check every two neighboring light rays. Each pair of light rays has the same phase difference of $2\pi(d\sin\theta\,/\lambda)$. When each pair of light rays have the phase difference as $m(2\pi)$. They will reach constructive interference. Thus we have the bright light at the condition of $d\sin\theta = m\lambda$.

X-Ray Diffraction

X-ray has a wavelength ranging from 0.01 to 10 nm. The lattice constants are about 0.2 - 1 nm.



Bruker D8 Discover X-Ray Diffraction System – light source is Cu Kα(1.540598 Å), electron transition from 2p to 1s orbitals

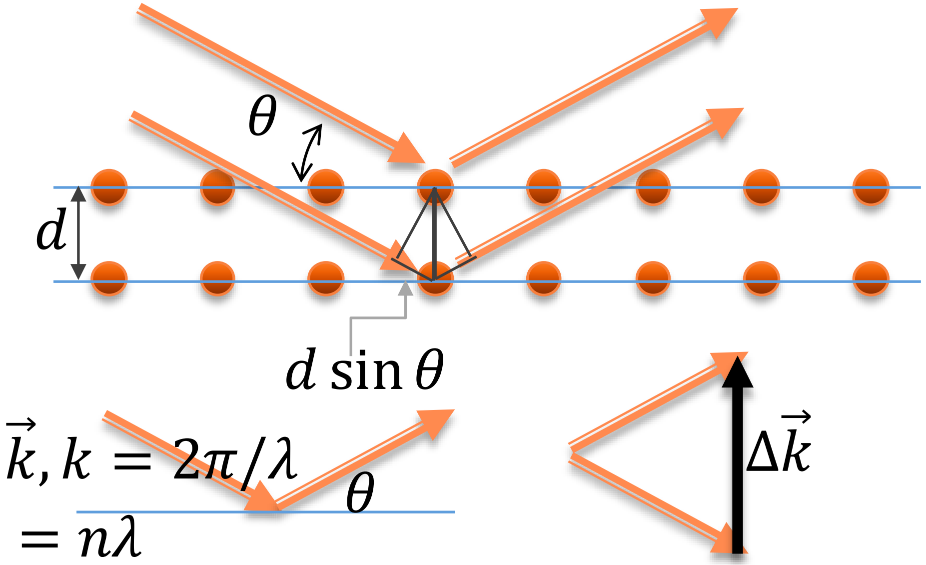
The Bragg Law:

$$2d\sin\theta = n\lambda$$

$$\left|\Delta\vec{k}\right| = 2k\sin\theta$$

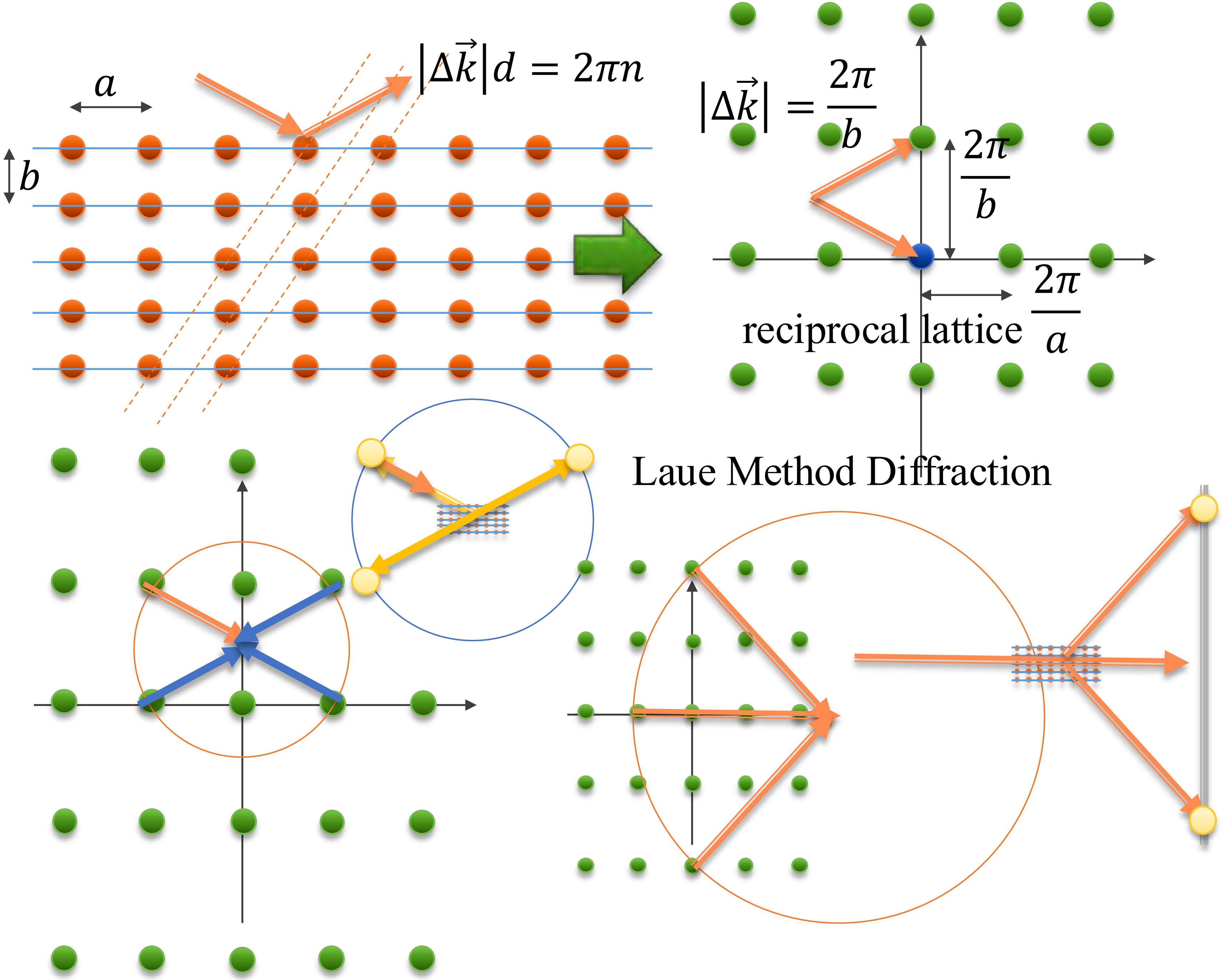
$$\left|\Delta\vec{k}\right|d = 2\pi n$$

$$2\frac{2\pi}{\lambda}\sin\theta\, d = 2\pi n \rightarrow 2d\sin\theta = n\lambda$$

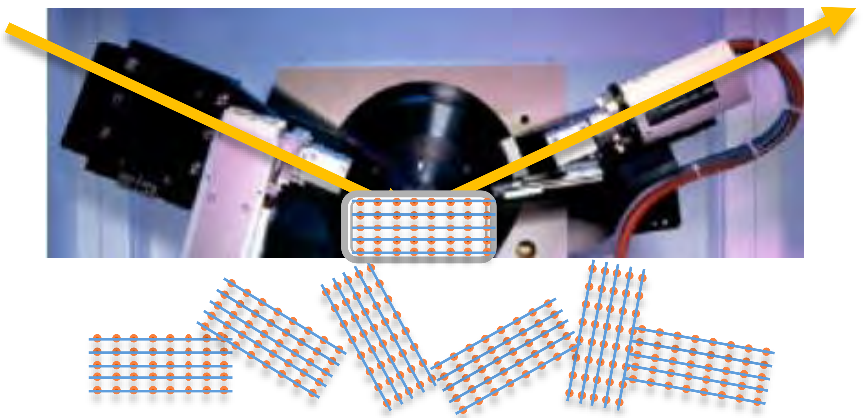


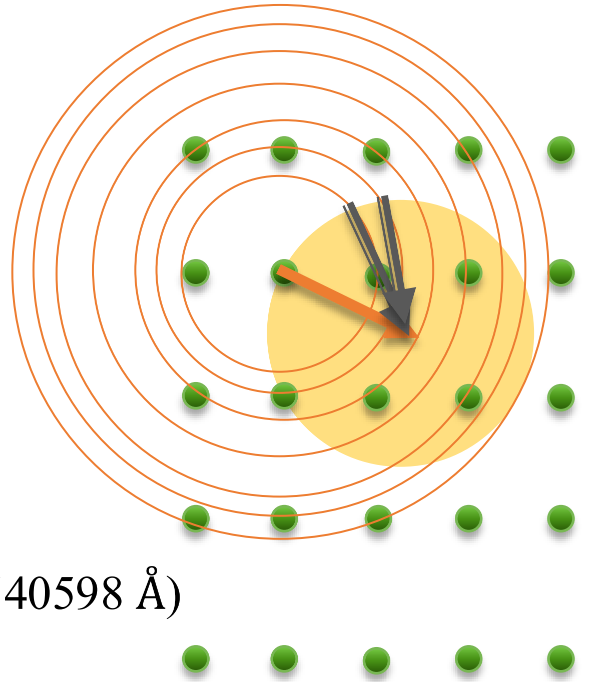
The change of light $\vec{k}$ vector multiplying the lattice distance gives $2\pi n$.

X-Ray Diffraction

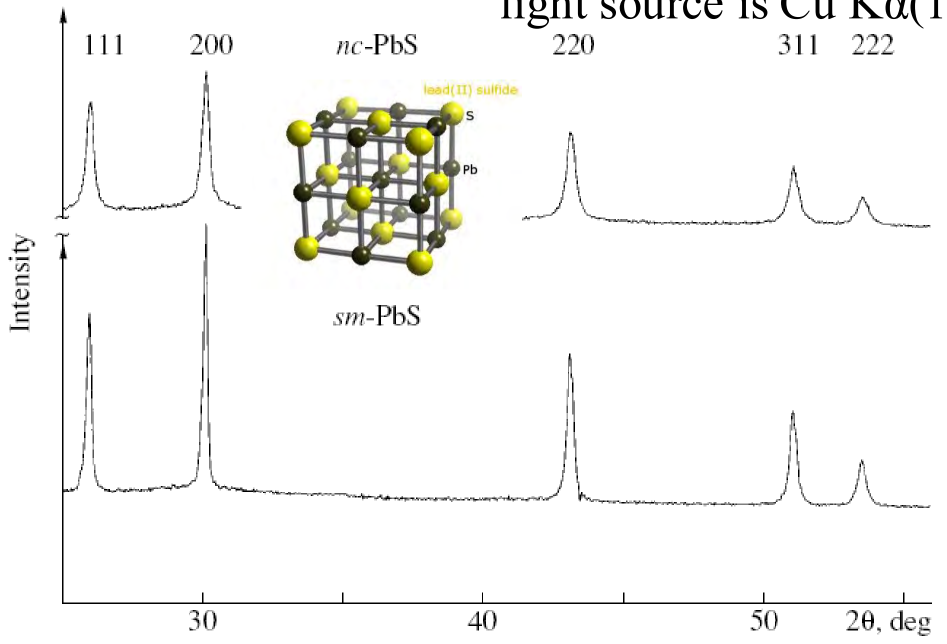


Powder X-Ray Diffraction





light source is Cu Kα(1.540598 Å)

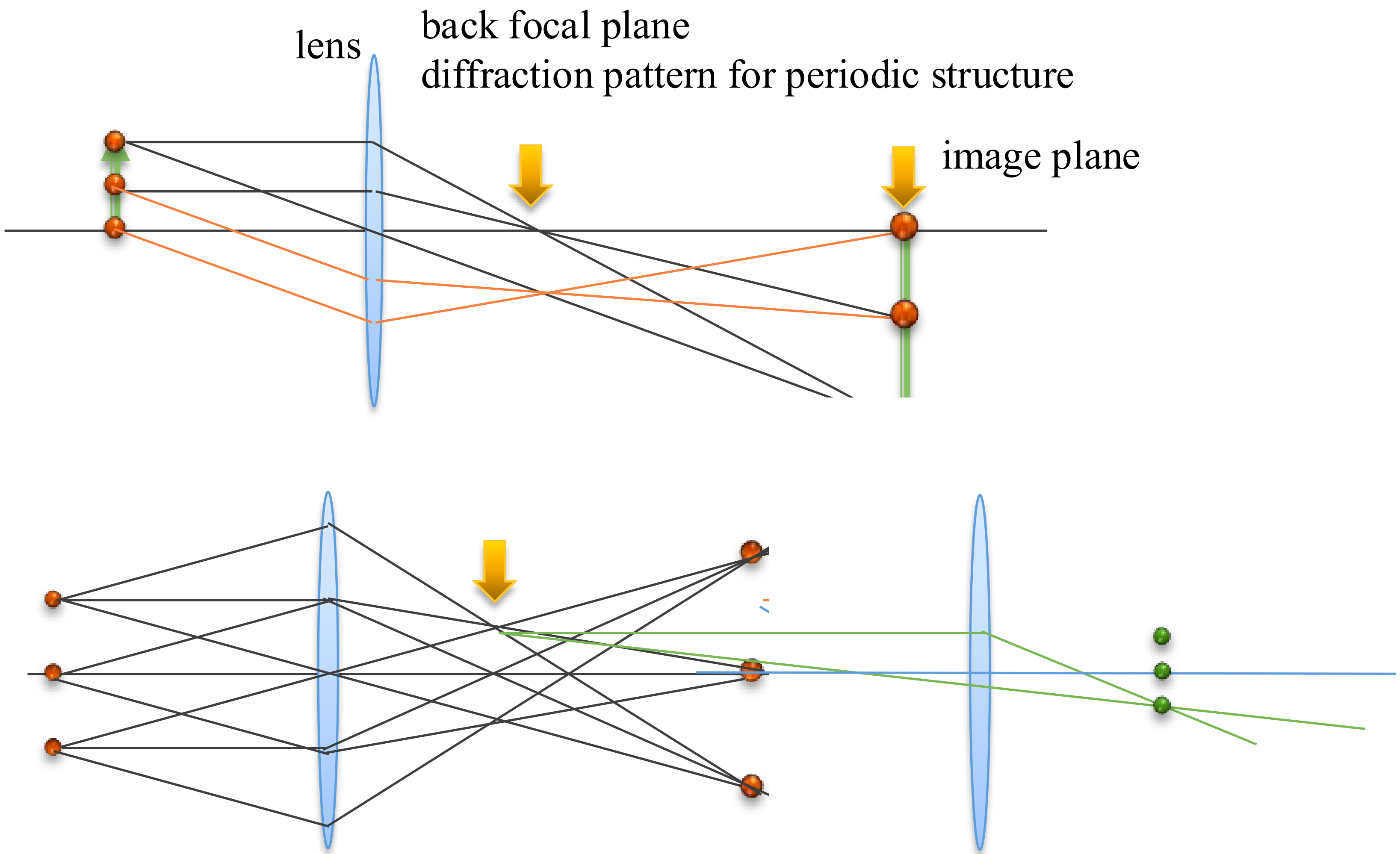


$$2d_{200}\sin(\pi/12) = 1.54$$

$$d_{200} = 2.98\text{Å}$$

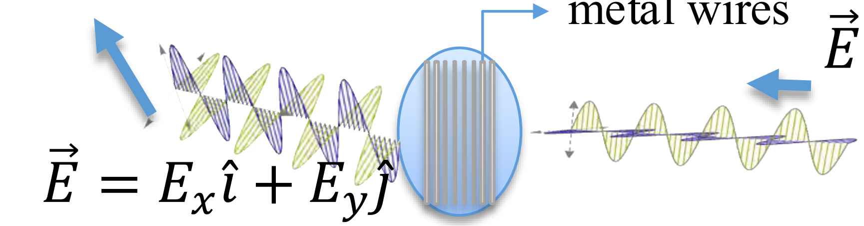
$$d_{100} = 5.95\text{Å}\ (5.936\text{Å})$$

Transmission Electron Microscope



Polarizer and Polarization of Light Waves by Selective Absorption/Reflection

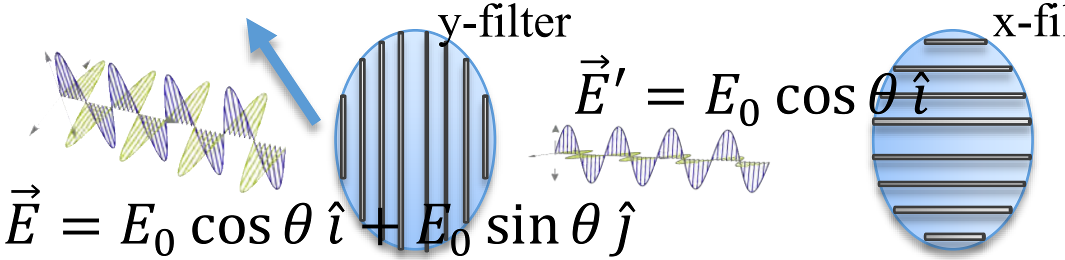
The wire-grid polarizer is made by many fine parallel metallic wires.



$\vec{E}' = E_x\hat{\imath}$

$\vec{E} = E_x\hat{\imath} + E_y\hat{\jmath}$

The component of the electric field along the axial direction of the metal wires will induce electrons' motion in the wires. The metal wires like metal reflect the light waves.

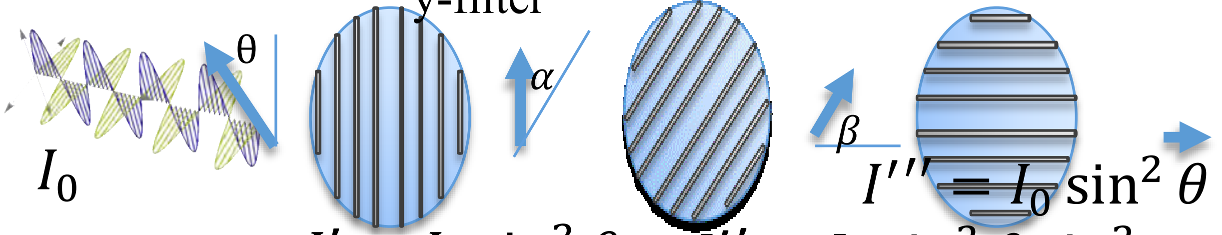


$\vec{E}' = E_0 \cos\theta\,\hat{\imath}$

$\vec{E}'' = I'' = 0$

$\vec{E} = E_0 \cos\theta\,\hat{\imath} + E_0 \sin\theta\,\hat{\jmath}$

$I_0 = \varepsilon_0 C\langle E_0^2(x,y,z,t)\rangle$ $\quad I' = \varepsilon_0 C\langle E_0^2(x,y,z,t)\cos^2\theta\rangle = I_0\cos^2\theta$



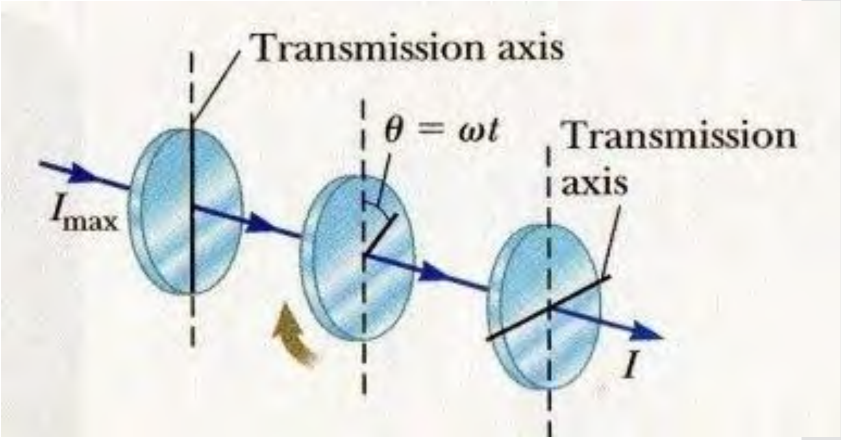
I_0

$I''' = I_0 \sin^2\theta \sin^2\alpha \sin^2\beta$

$I' = I_0 \sin^2\theta$ $\quad I'' = I_0 \sin^2\theta \sin^2\alpha$

Examples

In the right figure, suppose the transmission axes of the left and right polarizing disks are perpendicular to each other. Also, let the center disk be rotated on the common axis with an angular speed ω. If the unpolarized light is incident on the left disk with an intensity I_{max}, please calculate the beam emerging from the right disk.



After the 1st disk

$\frac{1}{2} I_{max}$ of light polarized in the y-axis

$$\vec{E} = E_0 \hat{y} \sin(kx - wt)$$

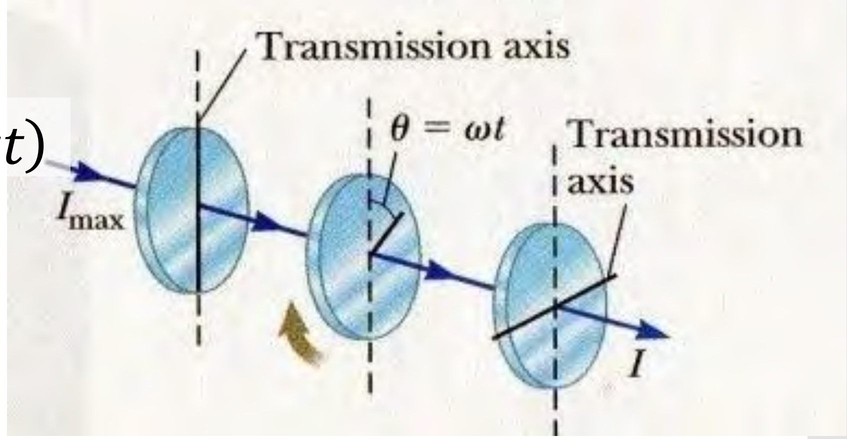
$$\frac{I_{max}}{2} \propto \langle E^2 \rangle = \frac{E_0^2}{2} \rightarrow I_{max} \propto E_0^2$$

After the 2nd disk

$$\vec{E} = E_0 \cos(\omega t)\, \hat{n} \sin(kx - wt)$$

Examples

In the right figure, suppose the transmission axes of the left and right polarizing disks are perpendicular to each other. Also, let the center disk be rotated on the common axis with an angular speed ω. If the unpolarized light is incident on the left disk with an intensity I_{max}, please calculate the beam emerging from the right disk.



After the 3rd disk

$$\vec{E} = E_0 \cos(\omega t) \sin(\omega t)\, \hat{x} \sin(kx - wt)$$

$$\langle E^2 \rangle = E_0^2 \sin^2(\omega t) \cos^2(\omega t) \frac{1}{2}$$

$$I_{max} \propto E_0^2$$

$$I = \frac{I_{max}}{2} \sin^2(\omega t) \cos^2(\omega t)$$

$$I = \frac{I_{max}}{8} \sin^2(2\omega t) = \frac{I_{max}}{16} (1 - \cos(4\omega t))$$

Polarization by Reflection - Brewster's Condition

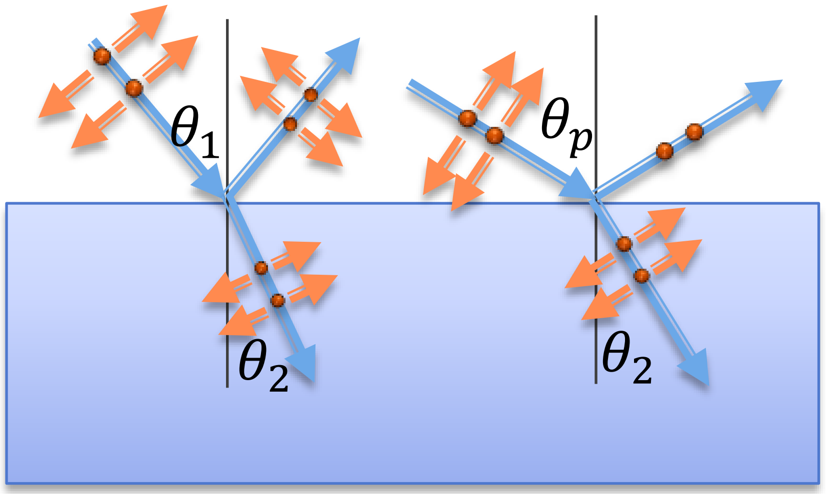
The polarization of the reflected light depends on the angle of incidence. If the angle of incidence is 0o, the reflected beam is unpolarized. For other angles, the reflected light is polarized to some extent. For the particular case of Brewster's condition, the reflected light is completely polarized.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$$\theta_p + \theta_2 = \frac{\pi}{2} \rightarrow \theta_2 = \frac{\pi}{2} - \theta_p$$

$$n_1 \sin\theta_p = n_2 \sin\left(\frac{\pi}{2} - \theta_p\right)$$

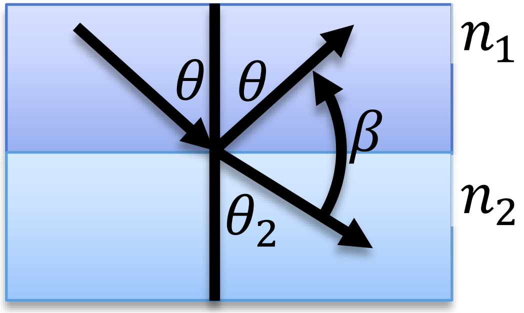
$$\tan\theta_p = \frac{n_2}{n_1}$$



Ref:

Examples

Light traveling in a medium of index of refraction n_1 is incident at an angle θ on the surface of a medium of index n_2. The angle between reflected and refracted ray is β. Please find the relation between θ and β.



$$n_1 \sin\theta = n_2 \sin\theta_2$$

$$\beta = \pi - \theta - \theta_2$$

$$\theta_2 = \pi - \theta - \beta$$

$$n_1 \sin\theta = n_2 \sin\theta_2 = n_2 \sin(\pi - \theta - \beta) = n_2 \sin(\theta + \beta)$$

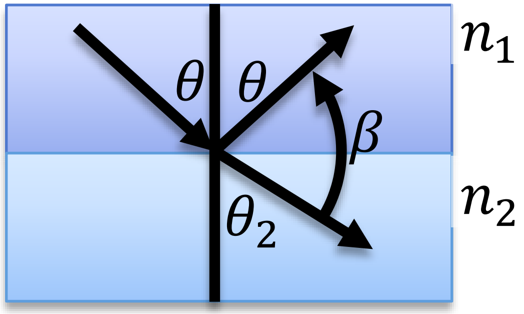
$$n_1 \sin\theta = n_2 \sin\theta \cos\beta + n_2 \cos\theta \sin\beta$$

$$(n_1 - n_2 \cos\beta) \sin\theta = n_2 \cos\theta \sin\beta$$

$$\tan\theta = \frac{n_2 \sin\beta}{n_1 - n_2 \cos\beta}$$

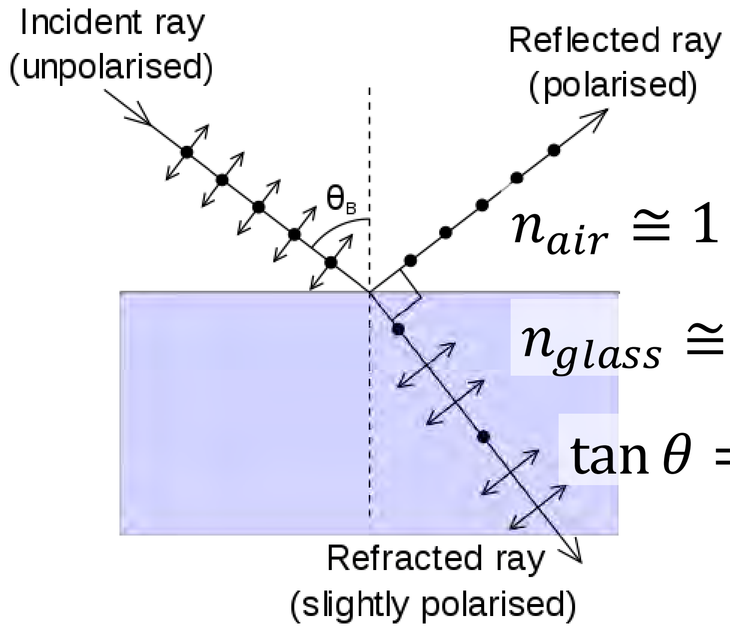
Examples

Light traveling in a medium of index of refraction n_1 is incident at an angle θ on the surface of a medium of index n_2. The angle between reflected and refracted ray is β. Please find the relation between θ and β.



If it fits to the Brewster's codition, $\beta = \frac{\pi}{2}$

$$\tan\theta = \frac{n_2}{n_1}$$



$n_{air} \cong 1$

$n_{glass} \cong 1.6$

$\tan\theta = 1.6 \rightarrow \theta \cong 58^o$

Examples

Two closely spaced wavelengths of light are incident on a diffraction grating. Starting with $d \sin\theta = m\lambda$, show that the angular dispersion of the grating is given by $\frac{d\theta}{d\lambda} = \frac{m}{d \cos\theta}$.

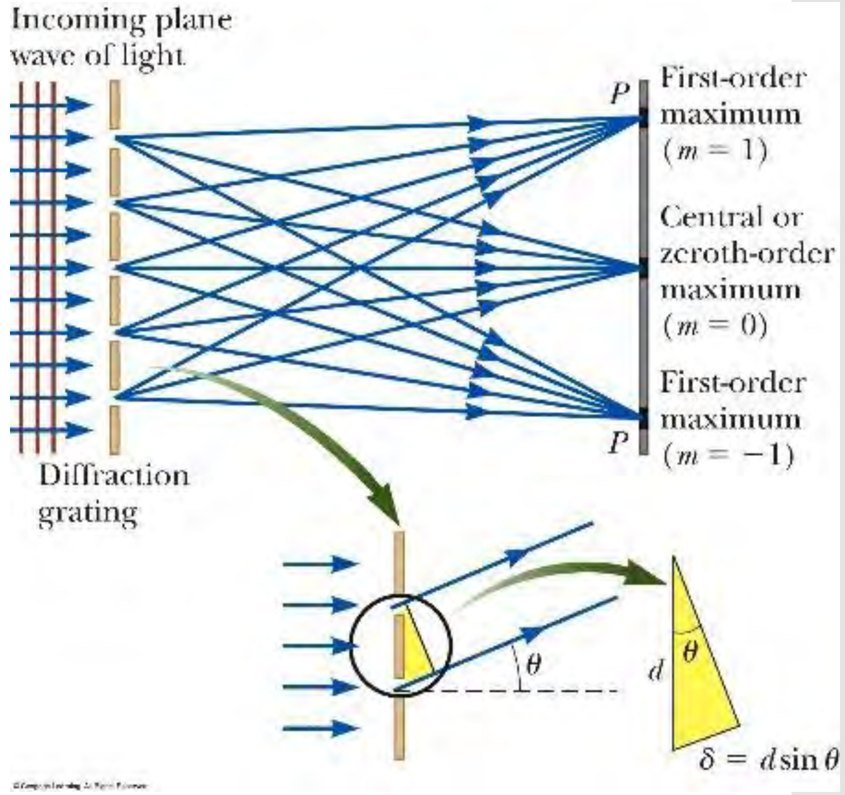
$$d \sin\theta = m\lambda$$

$$m = 0, \pm1, \pm2, \pm3$$

$$\sin\theta = \frac{m\lambda}{d}$$

$$\left(\frac{d}{d\lambda}\right)(\sin\theta) = \left(\frac{d}{d\lambda}\right)\left(\frac{m\lambda}{d}\right)$$

$$\cos\theta \frac{d\theta}{d\lambda} = \frac{m}{d} \rightarrow \frac{d\theta}{d\lambda} = \frac{m}{d\cos\theta}$$



Examples

In a single-slit diffraction experiment, the laser beam of wavelength 700 nm and the vertical slit of width 0.2 mm are used. The distance between the slit and the screen is 6 m. Please calculate the width of the central diffraction maximum on the screen.

The width of the central maximum is between the upper and lower first-dark fringes.

The upper dark fringe is estimated:

$$\frac{a}{2}\sin\theta = \frac{\lambda}{2} \rightarrow \frac{a}{2}\frac{y_{D1}}{L} = \frac{\lambda}{2} \rightarrow y_{D1} = \frac{\lambda L}{a} = \frac{(700 \times 10^{-9})(6)}{0.2 \times 10^{-3}}$$

$$y_{D1} = 0.021 \text{ m}$$

The width of the central maximum is $2y_{D1} = 0.042$ m.

Examples

Two slits of width a are separated by a distance d and are illuminated by light of wavelength λ. How many bright fringes are seen in the central diffraction maximum?

The width of the central diffraction is:

$$\frac{a}{2}\sin\theta = \frac{\lambda}{2} \rightarrow \frac{a}{2}\tan\theta = \frac{\lambda}{2} \rightarrow a\frac{y}{L} = \lambda \rightarrow y_{D,dark} = \frac{L\lambda}{a}$$

$$W = 2y_{D,dark} = \frac{2L\lambda}{a}$$

The constructive interference is at center and at the conditions of

$$d\sin\theta = \lambda \rightarrow d\frac{y}{L} = \lambda \rightarrow y_{I,bright} = \frac{\lambda L}{d}$$

The number of bright fringes is $2(y_D/y_I) - 1$ rather than $2(y_D/y_I) + 1$ because two bright fringes at the dark edge of diffraction turn dark.

$$N = 2\frac{L\lambda/a}{L\lambda/d} - 1 = 2\frac{d}{a} - 1$$

Ref:

Examples

Light of wavelength λ enters a human eye. The pupil is estimated to have a daytime diameter of D. (a) Estimate the limiting angle of resolution for the eye, assumes its resolution is limited by diffraction. (b) Determine the minimum separation distance d between two point sources that the eye can distinguish if the point sources are a distance L from the observer.

$$D \sin\theta_{min} = 1.22\lambda \rightarrow \theta_{min} \cong 1.22\frac{\lambda}{D}$$

$$\frac{d}{L} \cong \theta_{min} \rightarrow d \cong L\theta_{min} = 1.22\frac{L\lambda}{D}$$

Ref:

Examples

The diameter of the Keck Telescope at Mauna Kea, Hawaii, is 10 m. What is the resolution of the limiting angle for a light with wavelength of 600 nm?

The resolution regulation is $D\sin\theta_{min} = 1.22\lambda$.

The a very small angle approximation, $\sin\theta_{min} \cong \theta_{min}$, thus

$$\theta_{min} \cong \frac{1.22\lambda}{D} = 1.22\frac{600\times 10^{-9}}{10} = 7.32\times 10^{-8}\ \text{rad}$$