Chapter 27 Direct-Current Circuits

Physics II – Part I
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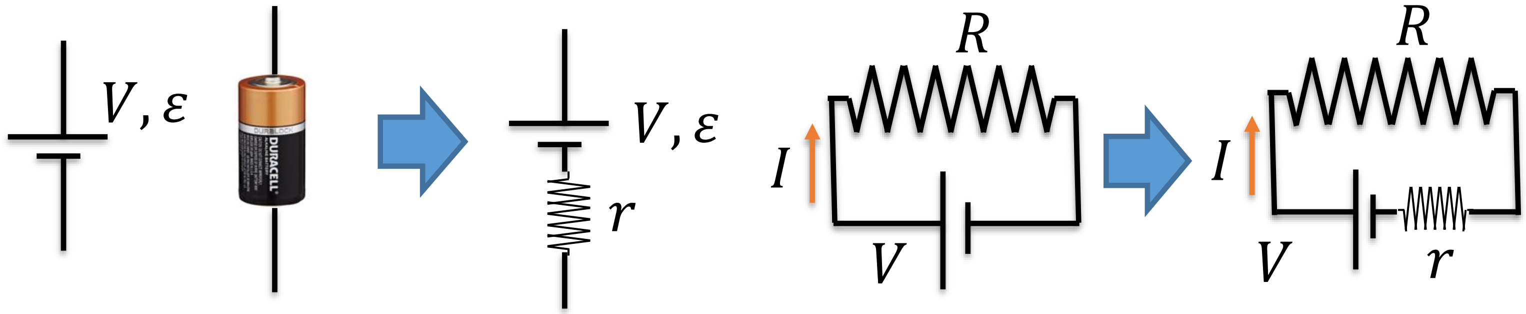
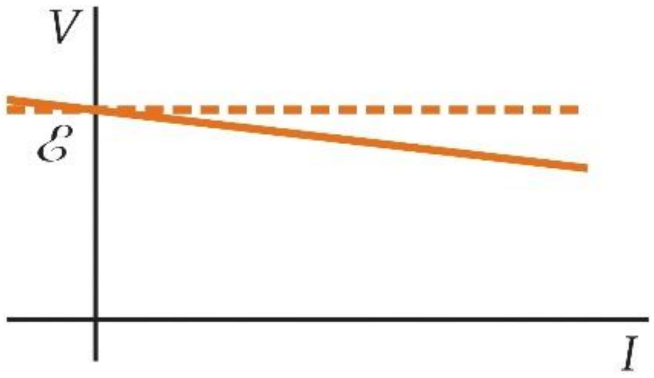
Element of DC Circuit - Battery

Electrochemical Cell, Galvanic Cell, Voltaic Cell



The electrochemical cells have internal resistors originating from chemical reactions and flow of ions.



The circuit current:

$$I = \frac{\varepsilon}{R + r}$$

The voltage across the resistor R is:

$$V = IR = \frac{R}{R + r}\varepsilon$$

The power exerted on the resistor R is $I \times \frac{R}{(R+r)}\varepsilon = \frac{R\varepsilon^2}{(R+r)^2}$.

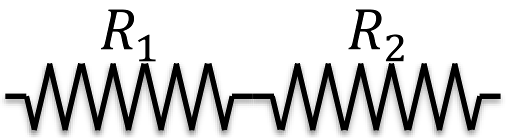
Resistors in Series or in Parallel Connection

Element of DC Circuit - Resistor

Connected in series:

The same current $I = I_1 = I_2$

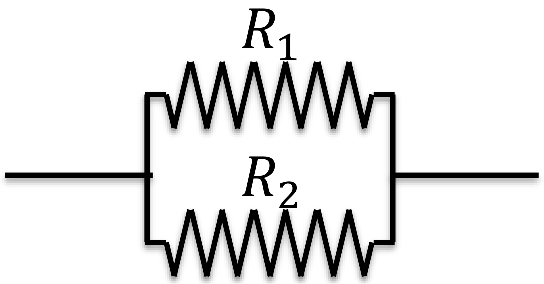
Sum the total voltage $V = V_1 + V_2$



$$V_1 = I_1 R_1, V_2 = I_2 R_2$$

$$V = V_1 + V_2 = I_1 R_1 + I_2 R_2 = I(R_1 + R_2)$$

$$R = \frac{V}{I} = R_1 + R_2$$



Connected in parallel:

The same voltage $V = V_1 = V_2$. Sum up the current $I = I_1 + I_2$

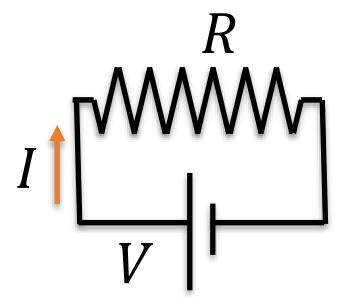
$$I = \frac{V_1}{R_1} + \frac{V_2}{R_2} = \frac{V}{R_1} + \frac{V}{R_2}$$

$$\frac{I}{V} = \frac{1}{R_1} + \frac{1}{R_2} \rightarrow \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Rules Applied in The Calculation of DC Circuits

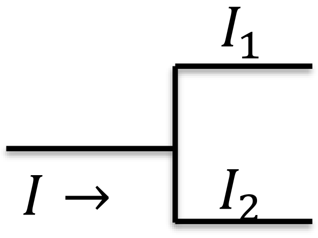
Kirchhoff's Rules

The algebraic sum of potential changes in a complete loop must be zero.



$$\oint \vec{E} \cdot d\vec{l} = 0 \quad \Rightarrow \quad V - IR = 0$$

At any branch point in the circuit, the current is conserved.



$$I = I_1 + I_2$$

Along the "loop direction", if the battery is placed from negative to positive polarity, the voltage is positively added.

Along the "loop direction", if the current is in the loop direction, the resistor gives a negative voltage IR.

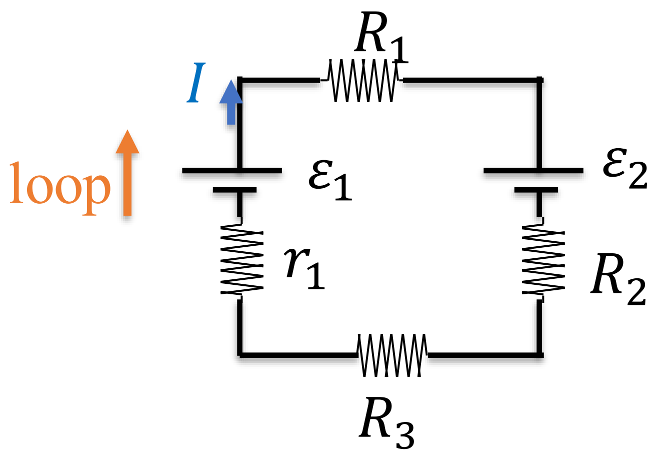
Rules Applied in The Calculation of DC Circuits

Kirchhoff's Rules

Single-Loop Circuit:

$$\varepsilon_1 - IR_1 - \varepsilon_2 - IR_2 - IR_3 - Ir_1 = 0$$

$$I = \frac{\varepsilon_1 - \varepsilon_2}{R_1 + R_2 + R_3 + r_1}$$



Multi-Loop Circuit:

$$I = I_1 + I_2$$

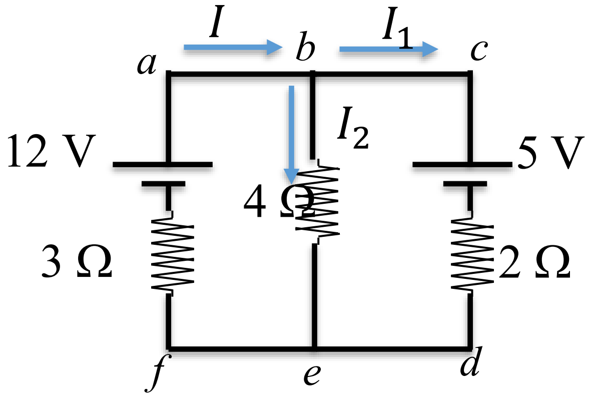
$12 - 4I_2 - 3I = 0$ --- *abef* loop

$12 - 5 - 2I_1 - 3I = 0$ --- *acdf* loop

$$12 - 4I_2 - 3(I_1 + I_2) = 0 \rightarrow 3I_1 + 7I_2 = 12$$

$$7 - 2I_1 - 3(I_1 + I_2) = 0 \rightarrow 5I_1 + 3I_2 = 7$$

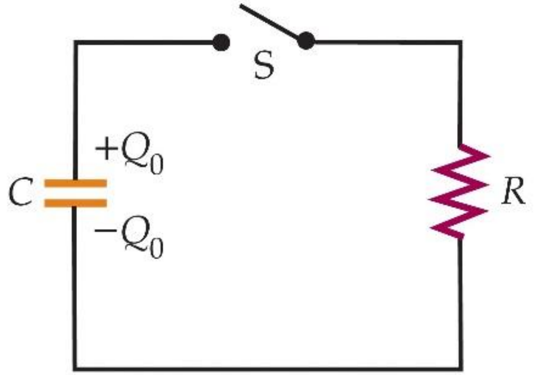
$$I_1 = \frac{1}{2}, I_2 = \frac{3}{2}, I = 2$$



Discharging a Capacitor

RC Circuits

A capacitor is charged and placed in the circuit as shown to the right figure. When the switch S is closed, please calculate the charge variation on the capacitor as a function of time.



Use Kirchhoff's rule: $-\dfrac{Q}{C} - IR = 0$

and the initial condition of $Q(0) = Q_0$

$$R\frac{dQ}{dt} + \frac{1}{C}Q = 0 \rightarrow \frac{dQ}{Q} = -\frac{dt}{RC} \rightarrow \int_{Q_0}^{Q(t')} \frac{dQ}{Q} = -\int_0^{t'} \frac{dt}{RC}$$

$$\ln(Q(t')/Q_0) = -t'/RC \ \rightarrow Q(t) = Q_0 e^{-t/RC}$$

$$I(t) = \left|\frac{dQ}{dt}\right| = \frac{Q_0}{RC} e^{-t/RC}$$

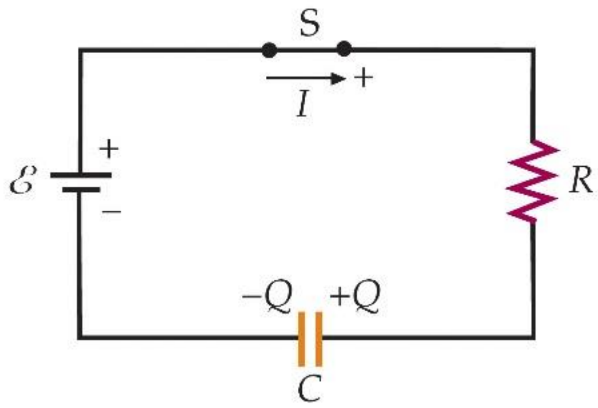
$\tau = RC$ is the time constant

Charging a Capacitor

RC Circuits

Use Kirchhoff's rule: $\varepsilon - \frac{Q}{C} - IR = 0$

and the initial condition of $Q(0) = 0$



$$R\frac{dQ}{dt} = \varepsilon - \frac{1}{C}Q \rightarrow \frac{dQ}{C\varepsilon - Q} = \frac{1}{RC}dt \rightarrow -\int_0^{Q(t')} \frac{d(C\varepsilon - Q)}{C\varepsilon - Q} = \int_0^{t'} \frac{dt}{RC}$$

$$[\ln(C\varepsilon - Q)]_0^Q = -\frac{t}{RC} \quad \ln\left(\frac{C\varepsilon - Q}{C\varepsilon}\right) = -\frac{t}{RC}$$

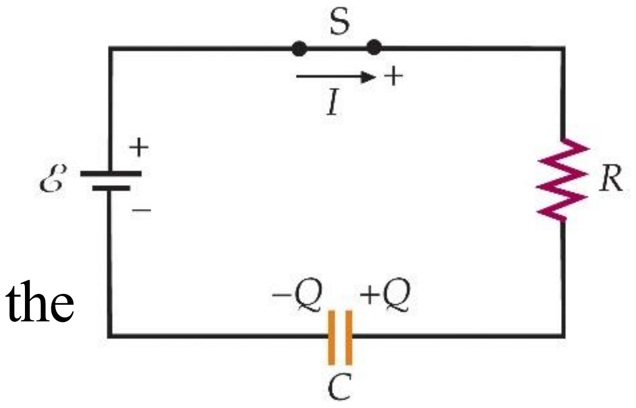
$$C\varepsilon - Q = C\varepsilon e^{-\frac{t}{RC}}$$

$$Q(t) = C\varepsilon\left(1 - e^{-\frac{t}{RC}}\right) \qquad I = \frac{dQ}{dt} = \frac{\varepsilon}{R}e^{-\frac{t}{RC}}$$

RC Circuits

Energy Conservation When Charging a Capacitor

When the battery push charge Q and $-Q$ accumulated on the capacitor, the electric potential energy consumed is $Q\varepsilon$.



The battery provided energy will be stored in the capacitor and consumed in the resistor.

The energy stored in the capacitor is $\int_0^Q V(q)dq = \int_0^Q \frac{q}{C}dq = \frac{Q^2}{2C} = \frac{1}{2}Q\varepsilon$.

The current in the loop is $I = \frac{\varepsilon}{R}e^{-\frac{t}{RC}}$.

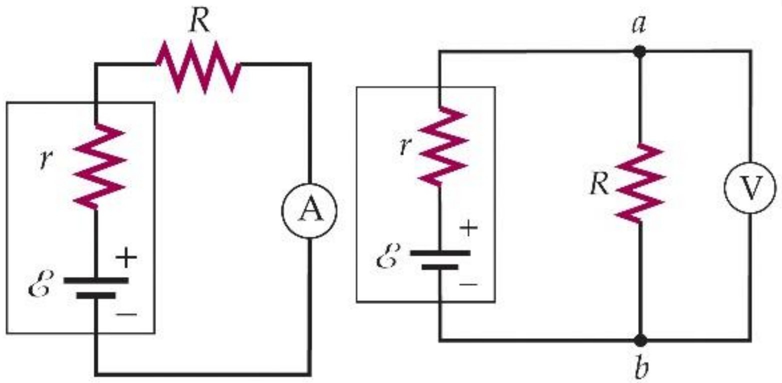
The total energy consumed in the resistor is

$$\int_0^\infty I^2 R dt = \frac{\varepsilon^2}{R}\int_0^\infty e^{-\frac{2t}{RC}}dt = \frac{C\varepsilon^2}{2}$$

Connected in Series or in Parallel?

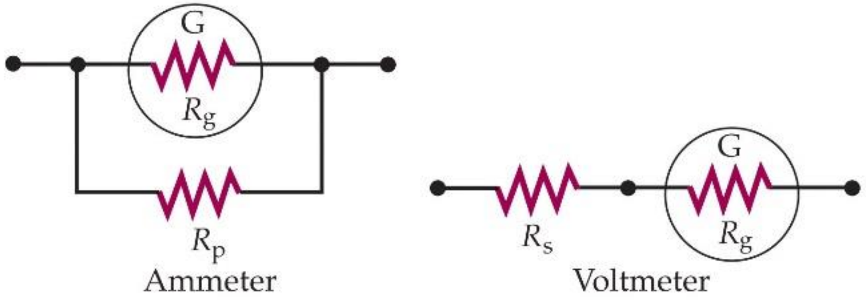
Electrical Meters

For voltage measurements, the meter is connected in parallel. For a current measurement, the meter is connected in series.



All meters are designed starting from the galvanometer. It is just a current in a loop that generates a magnetic field to attract a small piece of iron.

Using a galvanometer and a shunt resistance, we can make the ammeter and the voltmeter.



Internal Resistor in The Battery

Examples

For a battery of a given electromotive force ε and an internal resistance r, what value of external resistance R should be placed across the terminals to obtain the maximum power delivered to the resistor?

The current in the circuit is $I = \frac{\varepsilon}{R+r}$.

The power on the external resistor is $P(R) = I^2 R = \frac{\varepsilon^2 R}{(R+r)^2}$.

When the external resistor R is varied, what will be its value for a maximum output power?

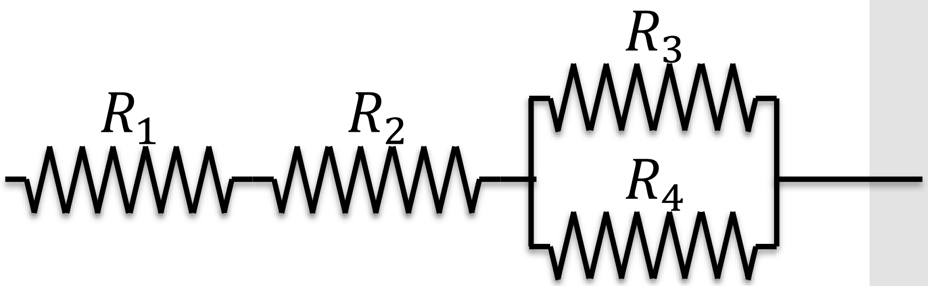
$$\frac{dP(R)}{dR} = 0 \rightarrow \frac{\varepsilon^2 (R+r)^2 - 2(R+r)\varepsilon^2 R}{(R+r)^4} = 0$$

$$R^2 + 2Rr + r^2 - 2R^2 - 2Rr = 0 \rightarrow r^2 - R^2 = 0 \rightarrow R = r$$

Equivalent Resistor

Examples

Find the equivalent resistance for the circuit shown in the figure.



At first, calculate the equivalent resistance for the combined R_3 and R_4 resistors.

$$\frac{1}{R_{34}} = \frac{1}{R_3} + \frac{1}{R_4} \rightarrow R_{34} = \frac{R_3 R_4}{R_3 + R_4}$$

Then, calculate the total resistance.

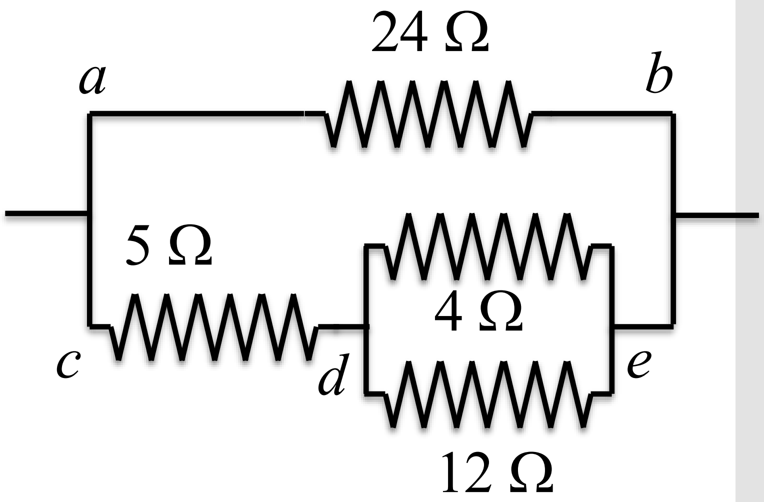
$$R_{total} = R_1 + R_2 + R_{34} = R_1 + R_2 + \frac{R_3 R_4}{R_3 + R_4}$$

Equivalent Resistor

Examples

Find the equivalent resistance for the circuit shown in the figure.

Use the rules of equivalent resistance for parallel or series connected resistors.



$$R_{de} = \frac{1}{\frac{1}{4} + \frac{1}{12}} = 3$$

$$R_{ce} = 5 + 3 = 8$$

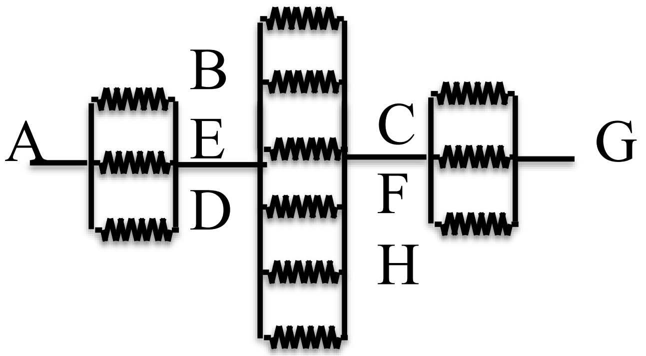
$$R_{total} = \frac{1}{\frac{1}{24} + \frac{1}{8}} = 6$$

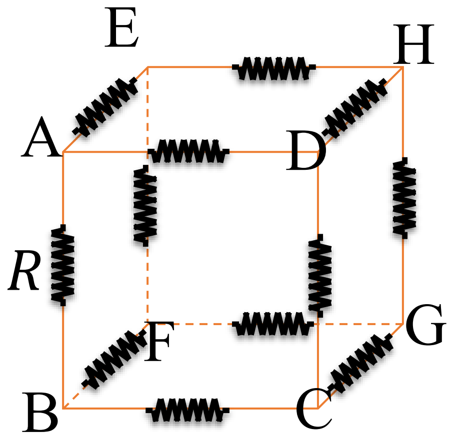
Ans: The equivalent resistance is 6 Ω.

Examples

Find the equivalent resistance for the circuit shown in the figure. Calculate R_{AG}. Assume that the resistance of each resistor is R.

The B, E, and D points are equipotential, and the F, C, and H points are equipotential.

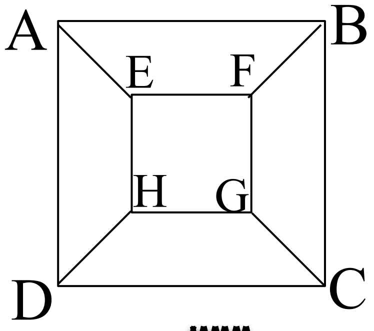


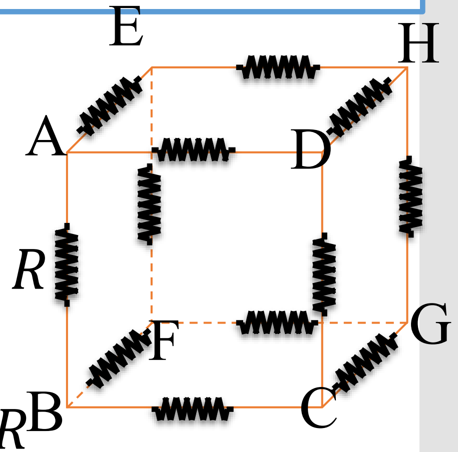


$$R_{eq} = \frac{R}{3} + \frac{R}{6} + \frac{R}{3} = \frac{5R}{6}$$

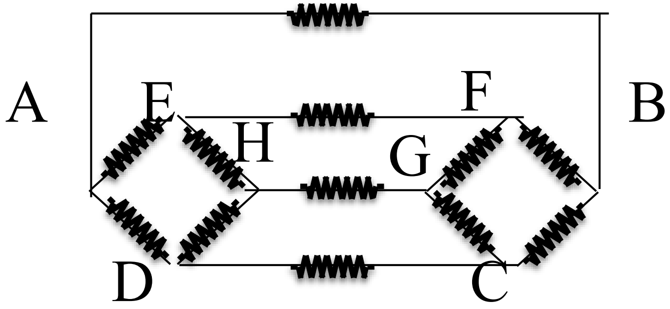
Examples

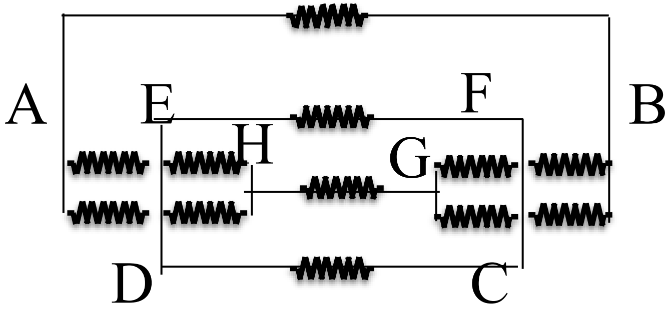
Find the equivalent resistance for the circuit shown in the figure. Calculate R_{AB}. Assume that the resistance of each resistor is R.





The points E and D are equipotential. The points F and C are equipotential.





$$R_{EH} = R_{GF} = R/2$$

$$R_{EHGF} = \frac{R}{2} + R + \frac{R}{2} = 2R$$

$$R_{EF} = \frac{1}{\frac{1}{R} + \frac{1}{2R} + \frac{1}{R}} = \frac{2}{5}R$$

$$E_{AEFB} = \frac{R}{2} + \frac{2R}{5} + \frac{R}{2} = \frac{7R}{5}$$

$$R_{AB} = \frac{1}{\frac{1}{R} + \frac{5}{7R}} = \frac{7R}{12}$$

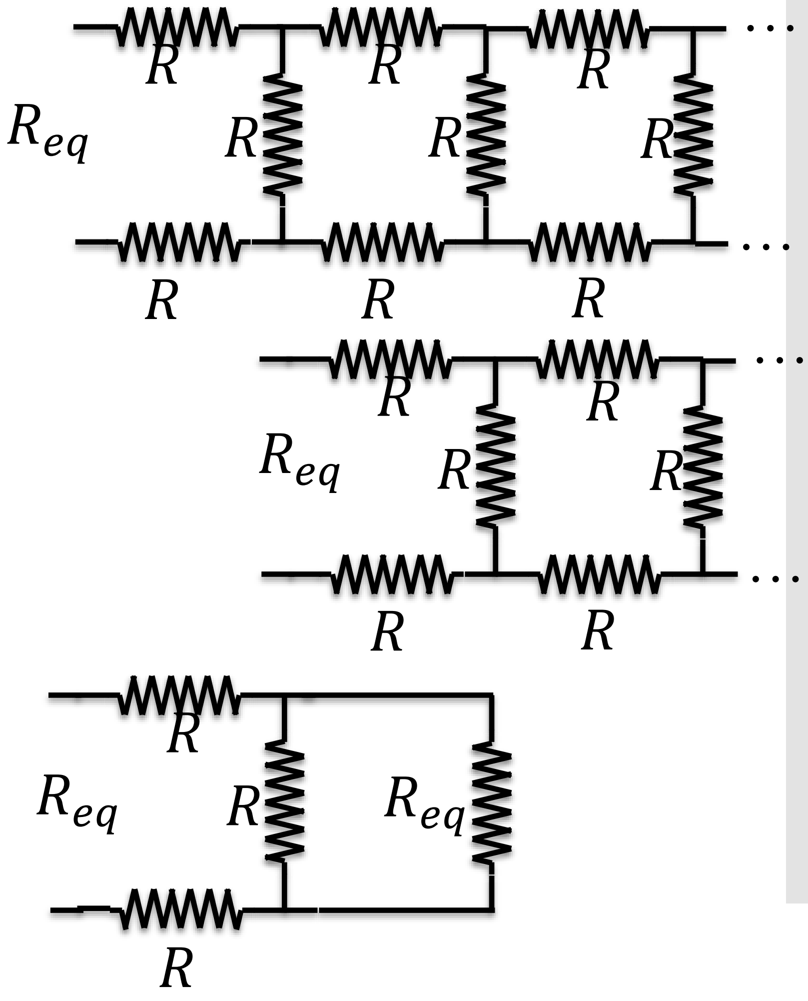
Examples

Find the equivalent resistance for the circuit shown in the figure.

$$R_{eq} = R + \frac{1}{\frac{1}{R} + \frac{1}{R_{eq}}} + R$$

$$R_{eq}^2 - 2RR_{eq} - 2R^2 = 0$$

$$R_{eq} = \left(1 + \sqrt{3}\right)R$$



Examples

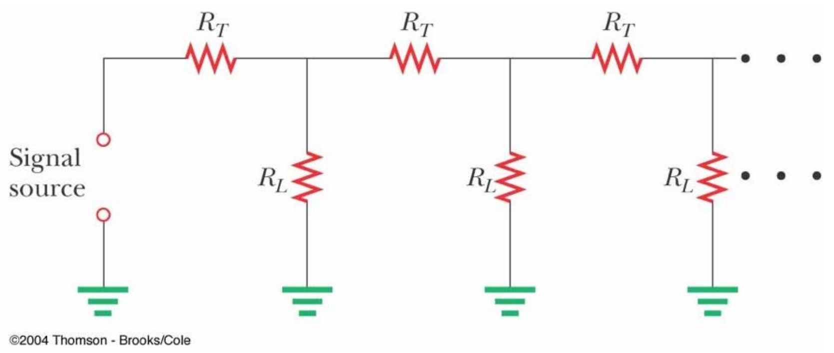
Please calculate the equivalent resistance of the circuit shown in the figure.

$$R_{eq} = R_T + \frac{1}{\frac{1}{R_L} + \frac{1}{R_{eq}}}$$

$$R_{eq} = R_T + \frac{R_L R_{eq}}{R_L + R_{eq}}$$

$$R_{eq}^2 - R_T R_{eq} - R_T R_L = 0$$

$$R_{eq} = \frac{R_T + \sqrt{R_T^2 + 4R_T R_L}}{2}$$



Knowledge to Keep You Safe

Examples

You are making a snack for some. You decide that coffee, toast, and popcorn would be a good start. You start the toaster and get some popcorn going in the microwave. Since your apartment is in an older building, you know you have problems with the **fuse blowing** when you turn too many things on. Should you start the coffeemaker? You look on the appliance and find that the toaster, the microwave, and the coffeemaker have ratings of 900 W, 1200 W, and 600 W, respectively. Past experiences have shown that your house has 20-A fuses and that the voltage is 120 V.

$$I_{toaster} = \frac{900}{120} = 7.5\ \text{A}$$

$$I_{m-wave} = \frac{1200}{120} = 10\ \text{A}$$

$$I_{c-maker} = \frac{600}{120} = 5\ \text{A}$$

$$7.5 + 10 + 5 = 22.5 > 20$$

When you turn on the coffeemaker, you blow your fuse and suddenly loose power. Without fuse protection, you may catch fire.

Examples

A voltage ΔV is applied to a series configuration of n resistors, each of resistance R. The circuit components are reconnected in parallel configuration, and voltage ΔV is again applied. Show that the power delivered to the series configuration is $\frac{1}{n^2}$ times the power delivered to the parallel configuration.

For a series connection, the equivalent resistance is $R_1 = nR$.

For a parallel connection, the equivalent resistance is $\frac{1}{R_2} = \frac{n}{R} \rightarrow R_2 = \frac{R}{n}$.

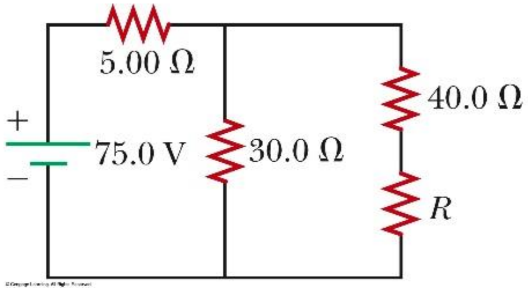
The power delivered to the series configuration is $P_1 = \frac{(\Delta V)^2}{R_1} = \frac{(\Delta V)^2}{nR}$.

The power delivered to the parallel configuration is $P_2 = \frac{(\Delta V)^2}{R_2} = \frac{(\Delta V)^2}{R/n} = n\frac{(\Delta V)^2}{R}$.

$$\frac{P_1}{P_2} = \frac{(\Delta V)^2/nR}{n(\Delta V)^2/R} = \frac{1}{n^2}$$

Examples

The resistor R in the right figure receives 20.0 W of power. Determine the value of R.



the current through R: $I_t = \dfrac{75}{5+\dfrac{1}{\dfrac{1}{30}+\dfrac{1}{40+R}}}$

$$I_t = \frac{75}{5+\dfrac{30(40+R)}{70+R}}$$

$$I_t = \frac{75(70+R)}{1550+35R}$$

$$I_R = \frac{30}{70+R}\frac{75(70+R)}{1550+35R} = \frac{2250}{1550+35R}$$

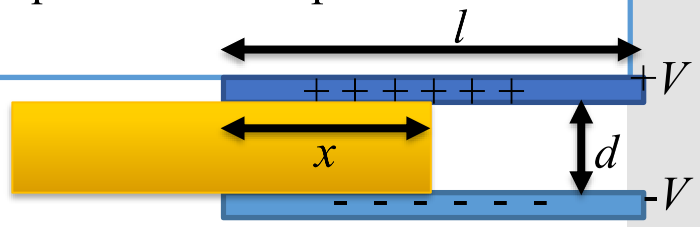
$$20 = \left(\frac{2250}{1550+35R}\right)^2 R$$

$$49R^2 - 5785R + 96100 = 0$$

$$R = 20\ or\ \frac{4805}{49}$$

Examples

A parallel-plate capacitor consists of square plates of edge l that are separated d apart. If the plates are maintained at a constant potential V and a square of dielectric slab of constant κ, area $A = l^2$, thickness d is inserted between the plates to a distance x as shown in the figure. Let σ_0 be the free charge density at the conductor-air surface. (a) Calculate the free charge density σ_κ at the capacitor-dielectric surface. (b) What is the effective capacitance? (c) What is the magnitude of the required force to prevent the dielectric slab from sliding into the plates?



Here the voltage V keeps constant. The charge distribution is not uniform. The charge density on the area with dielectric materials will be higher than that on the area without dielectric materials.

$$\sigma_\kappa = \kappa\sigma_0, \sigma_f = \sigma_0$$

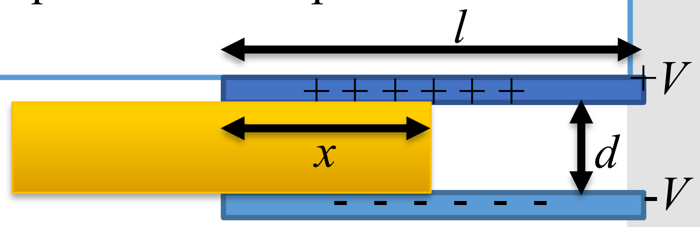
$$C_\kappa = \frac{lx\kappa\varepsilon_0}{d} \qquad C_f = \frac{l(l-x)\varepsilon_0}{d}$$

$$C_t = C_\kappa + C_f = \frac{lx\kappa\varepsilon_0}{d} + \frac{l(l-x)\varepsilon_0}{d}$$

$$C_t = \frac{l(l+x(\kappa-1))\varepsilon_0}{d}$$

Examples

A parallel-plate capacitor consists of square plates of edge l that are separated d apart. If the plates are maintained at a constant potential V and a square of dielectric slab of constant κ, area $A = l^2$, thickness d is inserted between the plates to a distance x as shown in the figure. Let σ_0 be the free charge density at the conductor-air surface. (a) Calculate the free charge density σ_κ at the capacitor-dielectric surface. (b) What is the effective capacitance? (c) What is the magnitude of the required force to prevent the dielectric slab from sliding into the plates?



$$U_C = \frac{CV^2}{2} \Rightarrow U_C = \frac{l(l + x(\kappa - 1))\varepsilon_0 V^2}{2d}$$

$$F_C = -\frac{dU}{dx} = -\frac{(\kappa - 1)l\varepsilon_0 V^2}{2d}$$

$$U_{\text{Env}} = QV \rightarrow U_{Sys} = -QV \qquad F_e = -\frac{dU_{Sys}}{dx} = Q\frac{dV}{dx} + V\frac{dQ}{dx} = V\frac{d(C_t V)}{dx}$$

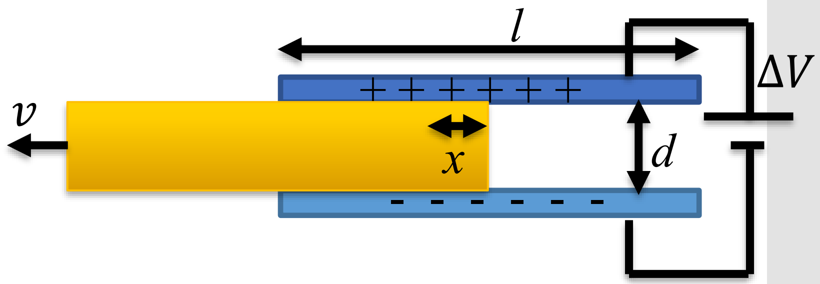
$$F_e = V^2\frac{d(C_t)}{dx} = \frac{(\kappa - 1)l\varepsilon_0 V^2}{d}$$

$$F = F_C + F_e = \frac{(\kappa - 1)l\varepsilon_0 V^2}{2d}$$

Examples

A parallel-plate capacitor consists of square plates of edge l that are separated d apart, where $d \ll l$. A potential difference ΔV is maintained between the plates. A material of dielectric constant κ fills half the space between the plates. The dielectric slab is withdrawn from the capacitor as shown in the figure. (a) Find the capacitance when the left edge of the dielectric is at a distance x from the center of the capacitor. (b) If the dielectric is removed at a constant speed v, what is the current in the circuit as the dielectric is being withdrawn?

$$C_t = C_\kappa + C_f = \frac{l(\frac{l}{2} - x)\kappa\varepsilon_0}{d} + \frac{l(\frac{l}{2} + x)\varepsilon_0}{d}$$



$$C_t = \frac{l\varepsilon_0(l(1+\kappa) + 2x(1-\kappa))}{2d}$$

$$Q = C_t\Delta V = \frac{l\varepsilon_0(l(1+\kappa) + 2x(1-\kappa))}{2d}\Delta V$$

$$\frac{dQ}{dt} = \left(\frac{d}{dx}\right)\left(\frac{l\varepsilon_0(l(1+\kappa) + 2x(1-\kappa))}{2d}\Delta V\right) \times \left(\frac{dx}{dt}\right)$$

$$\frac{dQ}{dt} = -\frac{l\varepsilon_0 v(\kappa - 1))}{d}\Delta V$$