Chapter 16
Wave Motion-I

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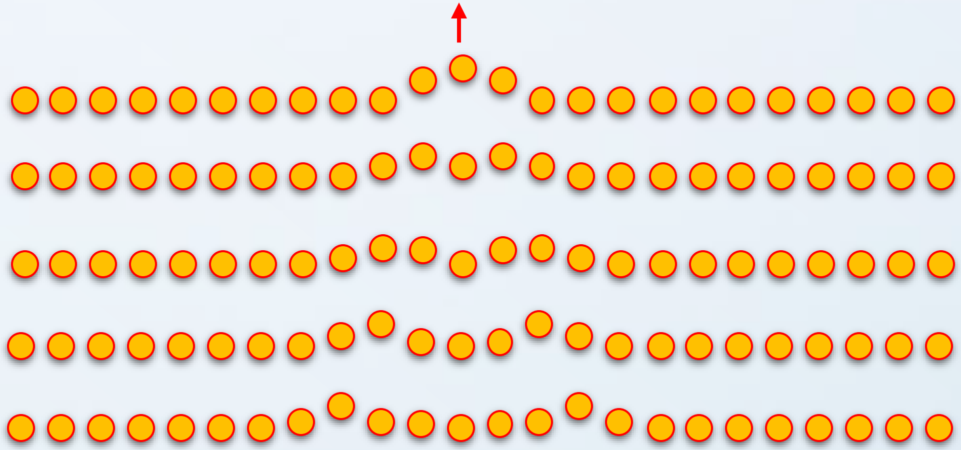
Outline

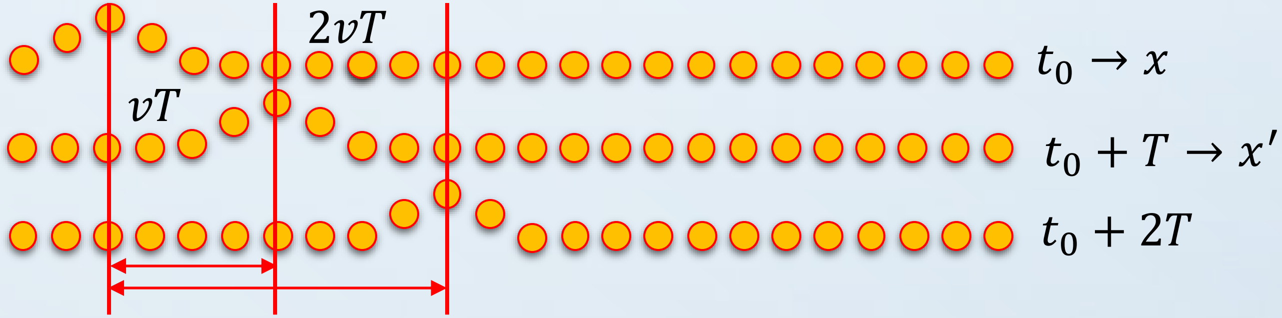
1. Propagation of a Disturbance
2. Traveling Wave Model
3. The Speed of Waves
4. Reflection and Transmission
5. Rate of Energy Transferred by Waves on Strings
6. The Wave Equation
7. Sound Waves
8. Speed of Sound Waves
9. Intensity of Sound Waves
10. The Doppler Effect

1. PROPAGATION OF A DISTURBANCE

The disturbance:



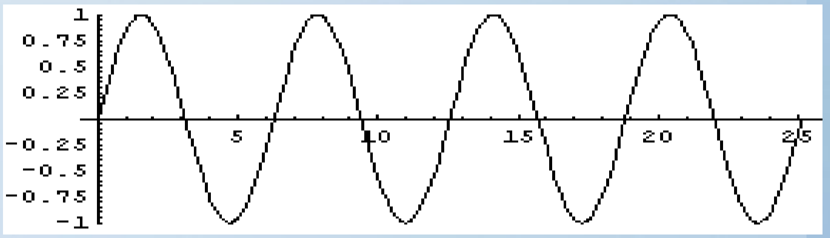
Traveling of the disturbance - wave



$$y(x, t_0) = f(x) \rightarrow y(x', t_0 + T) = y(x' - vT, t_0) = f(x' - vT)$$

$$y(x, t_0 + t)_{t_0=0} = f(x - vt) \ moving \ to \ the \ right$$

$$y(x, t_0 + t)_{t_0=0} = f(x + vt) \ moving \ to \ the \ left$$



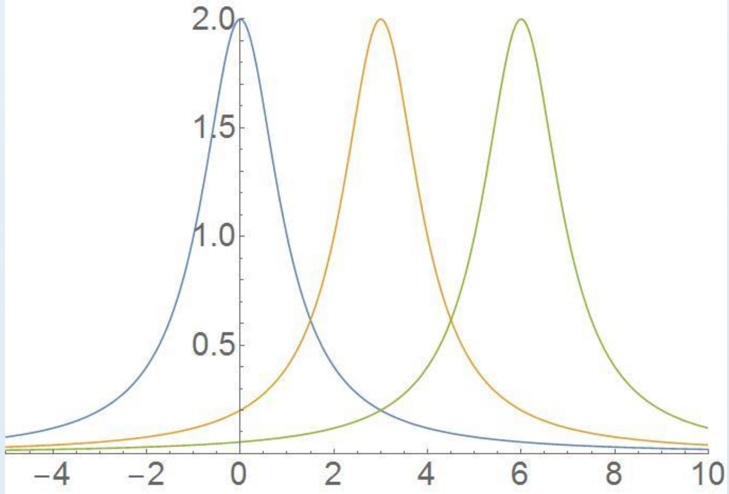
http://www.e-physics.net/Download/12wave/12wave.htm

1. PROPAGATION OF A DISTURBANCE

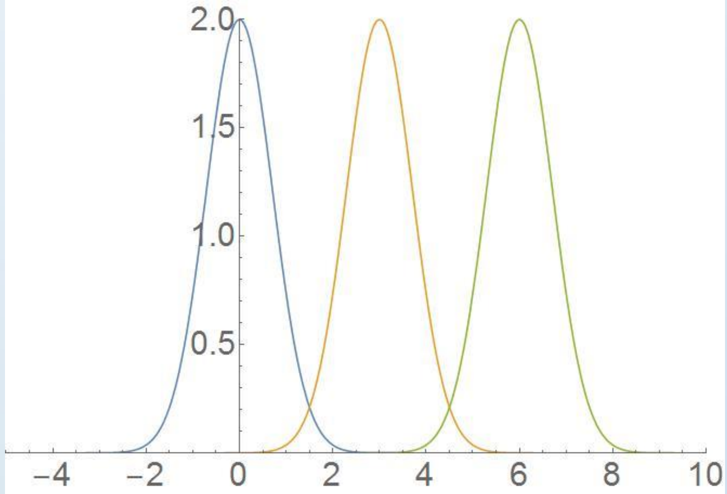
Example: A wave pulse moving to the right along the x-axis is represented by the wave function $y(x,t) = \frac{2.0}{(x-3.0t)^2+1}$ where x and y are measured in cm and t is in sec. Let us plot the wave form at t = 0, t = 1, and t = 2 s.

If $y(x,t) = f(x \pm vt)$, $y(x,t)$ is a wave function.

$y(x,t) = \frac{2.0}{(x-3.0t)^2+1}$, let $f(X) = \frac{2.0}{X^2+1}$, where $X = x - 3.0t$



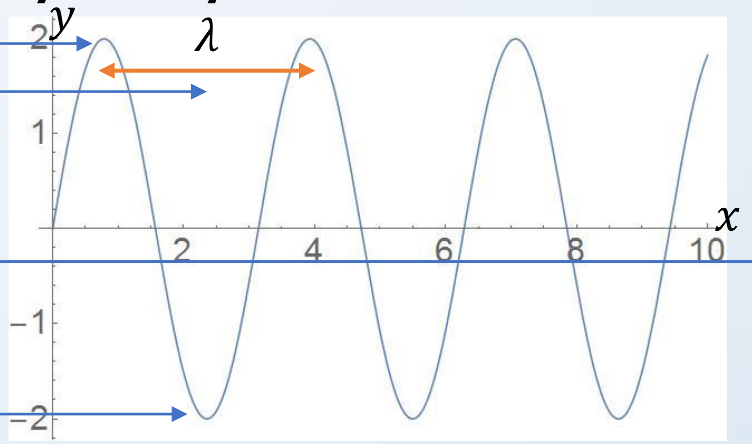
$y(x,t) = 2.0\exp(-(x-3.0t)^2)$

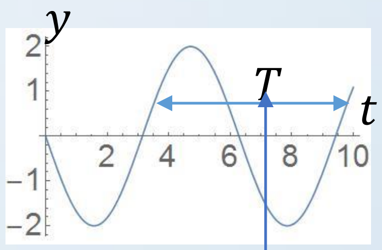


2. TRAVELING WAVE MODEL

The particles oscillate vertically in th y direction.

crest
trough
wave length λ
period T
frequency $f = 1/T$
wave speed: $v = f\lambda$





$$k = \frac{2\pi}{\lambda}, \omega = 2\pi f = \frac{2\pi}{T}, v = \frac{\omega}{k}$$

Start from an oscillator in the y direction at x=0: $\boldsymbol{y(0,t)} = \boldsymbol{y_0}\sin(\omega t)$

$y(0,t) = A\sin(\omega t) = A'\sin(-\omega t)$, for a wave traveling to the positive x direction

A wave function is expressed as $f(x - vt)$, match the two functions together at $x = 0$

$y(0,t) = A'\sin(-kvt) = A'\sin\big(k(0 - vt)\big)$

If $x \neq 0$, we have $y(x,t) = A'\sin\big(k(x - vt)\big) = A'\sin(kx - \omega t)$

2. TRAVELING WAVE MODEL

Example: A sinusoidal wave traveling in the positive x direction has an amplitude of 15 cm, a wavelength of 40 cm, and a frequency of 8.0 Hz. The vertical displacement of the medium at t = 0 and x = 0 is also 15 cm, (a) Find the angular wave number, period, angular frequency, and speed of the wave. (b) Determine the phase constant.

$$A = 15\ (cm), \lambda = 40\ (cm), f = 8.0\ (Hz)$$

(a)

$$k = \frac{2\pi}{\lambda} = 0.16\frac{rad}{cm}, T = \frac{1}{f} = 0.13\ s, \omega = 2\pi f = 50\frac{rad}{s}$$

$$v = f\lambda = 320\ cm/s$$

(b)

$$y(x,t) = A\sin(kx - \omega t + \phi) = 15\sin(0.16x - 50t + \phi)$$

$$y(0,0) = 15\sin(0 + \phi) = 15 \rightarrow \phi = \pi/2$$

3. THE SPEED OF WAVES

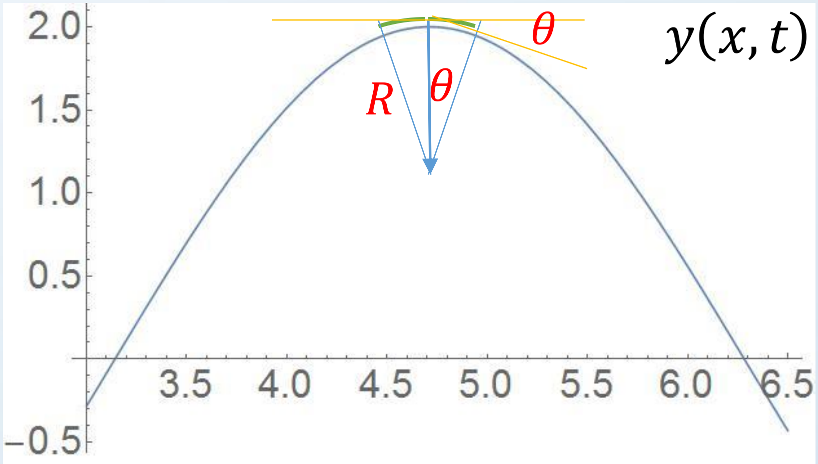
The oscillatory motion is a projection of a circular motion on the x-axis and the wave motion is the displacement away from equilibrium as a function of time and position.

$$F_r = 2T\sin\theta \cong 2T\theta$$

$$m = \mu\Delta s = \mu(2R\theta)$$

$$F_r = 2T\theta = ma = (2\mu R\theta)\frac{v^2}{R}$$

$$T = \mu v^2 \rightarrow v = \sqrt{\frac{T}{\mu}}$$



$$y(x,t) \rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

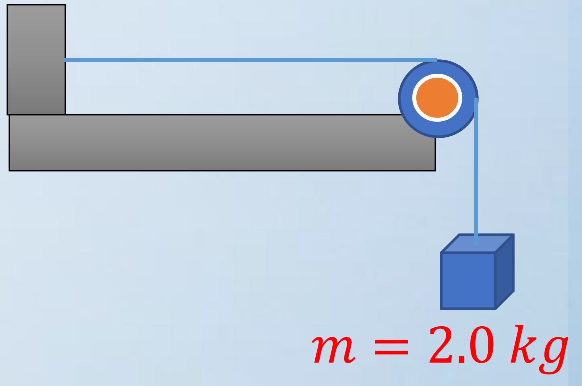
3. THE SPEED OF WAVES

Example: A uniform cord has a mass of 0.30 kg and a total length of 6.0 m. Tension is maintained in the cord by suspending an object of mass 2.0 kg from one end. Find the speed of a pulse on the cord. Assume that the tension is not affected by the mass of the cord.

$$m_{cord} = 0.30\,kg, L_{cord} = 6.0\,m$$

$$\mu = \frac{m}{L} = 0.050\,kg/m$$

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{2.0 \times 9.8}{0.050}} = 20\,m/s$$



3. THE SPEED OF WAVES

Guess the differential equation for the solutions of wave functions:

Start from the general wave function: $\boldsymbol{y(x,t)} = \boldsymbol{A}\sin(kx - \omega t)$

$$v_y = \frac{\partial y}{\partial t} = -\omega A\, cos(kx - \omega t)\,, a_y = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A \sin(kx - \omega t)$$

$$\frac{\partial y}{\partial x} = kA\, cos(kx - \omega t)\,, \frac{\partial^2 y}{\partial x^2} = -k^2 A \sin(kx - \omega t)$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2} \rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Example: Verify that the wave function $y(x,t) = \frac{2.0}{(x-3.0t)^2+1}$ is a solution to the linear wave equation.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{9}\frac{\partial^2 y}{\partial t^2}$$

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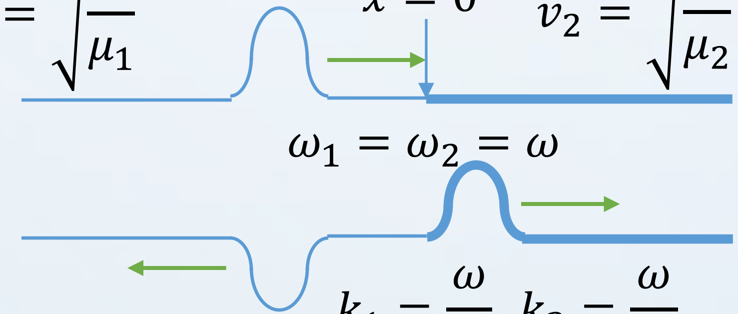
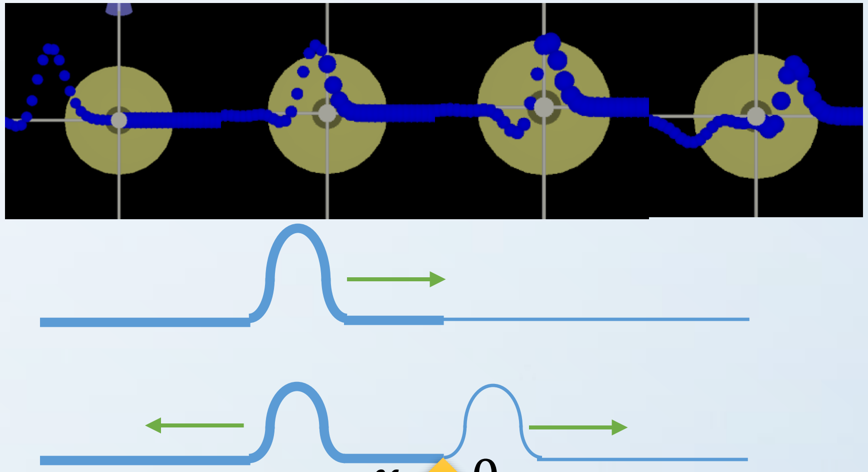
4. REFLECTION AND TRANSMISSION

$$v_1 = \sqrt{\frac{T}{\mu_1}} \qquad x = 0 \qquad v_2 = \sqrt{\frac{T}{\mu_2}}$$



$$\omega_1 = \omega_2 = \omega$$

$$k_1 = \frac{\omega}{v_1}, k_2 = \frac{\omega}{v_2}$$



$x = 0$

the displacement at the joint point

$$y_i(x,t) + y_r(x,t) = y_t(x,t)$$

the slope at the joint point:

$$\frac{d}{dx}\left(y_i(x,t) + y_r(x,t)\right) = \frac{d}{dx} y_t(x,t)$$

$$y_i(x,t) = A\cos(k_1 x - \omega t)$$

$$y_r(x,t) = B\cos(-k_1 x - \omega t)$$

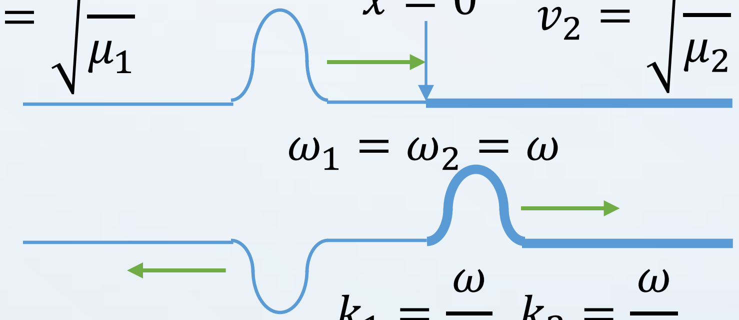
$$y_t(x,t) = C\cos(k_2 x - \omega t)$$

$$A\cos(k_1 x - \omega t) + B\cos(-k_1 x - \omega t) = C\cos(k_2 x - \omega t) \rightarrow A + B = C$$

$$-k_1 A\sin(k_1 x - \omega t) + k_1 B\sin(-k_1 x - \omega t) = -k_2 C\sin(k_2 x - \omega t) \rightarrow k_1 A - k_1 B = k_2 C$$

4. REFLECTION AND TRANSMISSION

$v_1 = \sqrt{\frac{T}{\mu_1}}$ $x = 0$ $v_2 = \sqrt{\frac{T}{\mu_2}}$



$\omega_1 = \omega_2 = \omega$

$k_1 = \frac{\omega}{v_1}, k_2 = \frac{\omega}{v_2}$

$$y_i(x,t) = A\cos(k_1 x - \omega t)$$

$$y_r(x,t) = B\cos(-k_1 x - \omega t)$$

$$y_t(x,t) = C\cos(k_2 x - \omega t)$$

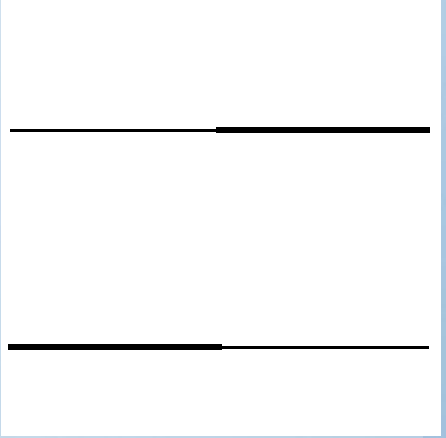
$$x = 0 \rightarrow A + B = C, k_1 A - k_1 B = k_2 C$$

$$C - B = A$$

$$k_2 C + k_1 B = k_1 A$$

$$B = \frac{k_1 - k_2}{k_1 + k_2} A, C = \frac{2k_1}{k_1 + k_2} A, if\ v_1 > v_2, k_1 < k_2 \ \& \ B < 0$$

$$y_r(x,t) = -|B|\cos(-k_1 x - \omega t) = |B|\cos(-k_1 x - \omega t + \pi)$$



http://www.e-physics.net/Download/12wave/12wave.htm

5. RATE OF ENERGY TRANSFERRED BY WAVES ON STRINGS

Kinetic energy in a period

$$y(x,t) = A\sin(kx - \omega t)\,,\, v_y = -\omega A\,cos(kx - \omega t)\,, \frac{\partial y}{\partial x} = kA\,cos(kx - \omega t)$$

$$dK = \frac{1}{2}(dm)v_y^2 = \frac{1}{2}(\mu dx)v_y^2 = \frac{1}{2}\mu\omega^2 A^2\,cos^2(kx - \omega t)\,dx$$

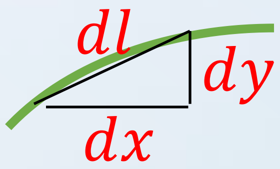
at t=0, the average kinetic energy in a period of length is

$$K = \int_0^\lambda dK = \int_0^\lambda \frac{1}{2}\mu\omega^2 A^2\,cos^2(kx)\,dx$$

$$\int_0^\lambda \cos^2\left(\frac{2\pi}{\lambda}x\right)dx = \int_0^\lambda \frac{1 + \cos\left(\frac{4\pi}{\lambda}x\right)}{2}dx = \frac{\lambda}{2}$$

$$K = \int_0^\lambda \frac{1}{2}\mu\omega^2 A^2\,cos^2(kx)\,dx = \frac{1}{4}\mu\omega^2 A^2\lambda$$

5. RATE OF ENERGY TRANSFERRED BY WAVES ON STRINGS



Potential energy in a period

$$y(x,t) = A\sin(kx - \omega t)\,,\, v_y = -\omega A\cos(kx - \omega t)\,,\frac{\partial y}{\partial x} = kA\cos(kx - \omega t)$$

$$dU = T(dl - dx) = T\left(\sqrt{(dx)^2 + (dy)^2} - dx\right)$$

$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (\partial y/\partial x)^2}dx \qquad f(x) = \sqrt{1+x} \rightarrow f(x) \cong f(0) + \frac{f'(0)}{1!}x = 1 + \frac{x}{2}$$

$$dU = T\left(\sqrt{(dx)^2 + (dy)^2} - dx\right) \cong T\left(\left(1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right)(dx) - dx\right) = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx$$

$$U = \int_0^\lambda \frac{1}{2}Tk^2A^2\cos^2(kx)\,dx = \frac{1}{4}\mu v^2k^2A^2\lambda = \frac{1}{4}\mu\omega^2A^2\lambda$$

Total energy in a period & transferred power

$$E = \frac{1}{2}\mu\omega^2A^2\lambda \rightarrow P = \frac{E}{T} = \frac{1}{2}\mu\omega^2A^2v = \frac{1}{2}\mu v_{ROT}^2 v$$

5. RATE OF ENERGY TRANSFERRED BY WAVES ON STRINGS

Example: A string with linear mass density 5.0 x 10^{-2} kg/m is under a tension of 80 N. How much power must be supplied to the string to generate sinusoidal waves at a frequency of 60 Hz and an amplitude of 6.0 cm?

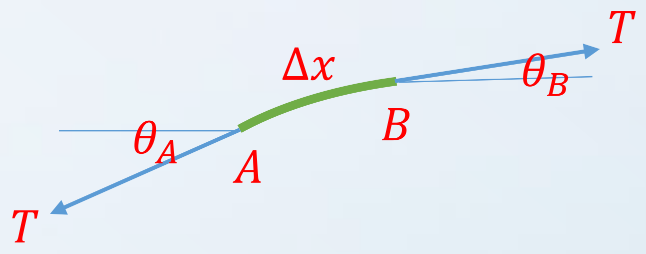
$$\mu = 0.050\frac{kg}{m}, f = 60\ Hz, T = 80\ N, A = 0.060\ m$$

$$v = \sqrt{\frac{80}{0.050}} = 40\ m/s$$

$$P = \frac{E}{T} = \frac{1}{2}\mu\omega^2 A^2 v = \frac{1}{2}(0.050)(2\pi \times 60)^2(0.060)^2(40) = 510\ W$$

6. THE WAVE EQUATION

Derive the wave equation:



$$F_{net} = T\sin\theta_B - T\sin\theta_A$$

$$F_{net} \cong T\tan\theta_B - T\tan\theta_A$$

$$F_{net} = T\left(\frac{\partial y}{\partial x}\right)_B - T\left(\frac{\partial y}{\partial x}\right)_A = T\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\Delta x$$

$$F_{net} = ma = (\mu\Delta x)\left(\frac{\partial^2 y}{\partial t^2}\right) \rightarrow (\mu\Delta x)\left(\frac{\partial^2 y}{\partial t^2}\right) = T\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\Delta x$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{\frac{T}{\mu}}\frac{\partial^2 y}{\partial t^2} \rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

EXERCISE

(a) Show that the function $y(x,t) = x^2 + v^2t^2$ is the solution to the wave equation of $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$. (b) Show that this function can be written as $f(x+vt) + g(x-vt)$ and determine the function forms for f and g.

(a)

$$\frac{\partial}{\partial x}(x^2 + v^2t^2) = 2x, \frac{\partial^2}{\partial x^2}(x^2 + v^2t^2) = 2$$

$$\frac{\partial}{\partial t}(x^2 + v^2t^2) = 2v^2t, \frac{\partial^2}{\partial t^2}(x^2 + v^2t^2) = 2v^2$$

$$\frac{\partial^2 y(x,y)}{\partial x^2} = 2 = \frac{\partial^2 y(x,y)}{\partial t^2}\frac{1}{v^2} \rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

(b)

$$x^2 + v^2t^2 = \frac{1}{2}\left((x+vt)^2 + (x-vt)^2\right)$$

$$x^2 + v^2t^2 = f(x+vt) + g(x-vt)$$

$$\rightarrow f(x) = \frac{x^2}{2}, g(x) = \frac{x^2}{2}$$

EXERCISE

Assume an object of mass M is suspended from the bottom of the rope of mass m and length L. (a) Show that the time interval for a transverse pulse to travel the length of the rope is $\Delta t = 2\sqrt{\frac{L}{mg}}\left(\sqrt{M+m} - \sqrt{M}\right)$. (b) Show that for $m \ll M$, the expression in part (a) reduces to $\Delta t = \sqrt{\frac{mL}{Mg}}$.

(a)

$$dt = \frac{dx}{v}, v = \sqrt{\frac{T}{\lambda}}$$

$$T = Mg + \lambda x g, \lambda = \frac{m}{L}$$

$$dt = \frac{dx}{\sqrt{\left(Mg + \frac{mx}{L}g\right)/(m/L)}} = \frac{dx}{\sqrt{xg + \frac{MgL}{m}}}$$

$$\Delta t = \frac{1}{\sqrt{g}}\int_0^L \frac{dx}{\sqrt{x + \frac{ML}{m}}} = \left[\frac{2}{\sqrt{g}}\sqrt{x + \frac{ML}{m}}\right]_{x=0}^{x=L}$$

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(a)

$$\Delta t = \frac{2}{\sqrt{g}}\left(\sqrt{L + \frac{ML}{m}} - \sqrt{\frac{ML}{m}}\right) = 2\sqrt{\frac{L}{mg}}\left(\sqrt{M+m} - \sqrt{M}\right)$$

(b)

$$\Delta t = 2\sqrt{\frac{L}{mg}}\left(\sqrt{M+m} - \sqrt{M}\right) = 2\sqrt{\frac{L}{mg}}\left(\sqrt{M}\sqrt{1+\frac{m}{M}} - \sqrt{M}\right)$$

$$m \ll M \rightarrow \Delta t \cong 2\sqrt{\frac{L}{mg}}\left(\sqrt{M}\left(1 + \frac{1}{2}\frac{m}{M}\right) - \sqrt{M}\right)$$

$$\Delta t \cong 2\sqrt{\frac{L}{mg}\frac{1}{2}\frac{m}{M}}\sqrt{M} = \sqrt{\frac{mL}{Mg}}$$

EXERCISE

A rope of total mass m and length L is suspended vertically. A pulse travels from the bottom to the top of the rope in an approximate time interval $\Delta t = 2\sqrt{L/g}$ with a speed that varies with position x measured from the bottom of the rope as $v = \sqrt{gx}$. Assume the linear wave equation describes waves at all locations on the rope.

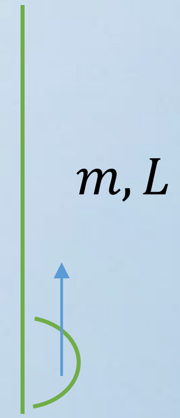
(a) Over what time interval does a pulse travel half-way up the rope? Give your answer as a fraction of $2\sqrt{L/g}$.

(b) A pulse starts traveling up the rope. How far has it travels at a time interval $\sqrt{L/g}$.

$$v = \sqrt{\frac{T}{\lambda}} = \sqrt{\frac{\lambda x g}{\lambda}} = \sqrt{xg}$$

$$dt = \frac{dx}{v} = \frac{1}{\sqrt{g}}\frac{dx}{\sqrt{x}}$$

$$\Delta t = \frac{1}{\sqrt{g}}\int_0^L \frac{dx}{\sqrt{x}} = \frac{1}{\sqrt{g}}\left[2\sqrt{x}\right]_{x=0}^{x=L} = 2\sqrt{\frac{L}{g}}$$



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(a) Over what time interval does a pulse travel half-way up the rope? Give your answer as a fraction of $2\sqrt{L/g}$.
(b) A pulse starts traveling up the rope. How far has it travels at a time interval $\sqrt{L/g}$.

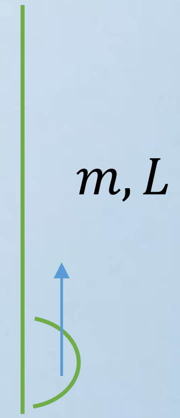
(a)

$$\Delta t_{L/2} = \frac{1}{\sqrt{g}}\int_0^{L/2}\frac{dx}{\sqrt{x}} = \frac{1}{\sqrt{g}}[2\sqrt{x}]_{x=0}^{x=L/2} = 2\sqrt{\frac{L/2}{g}} = \frac{1}{\sqrt{2}}2\sqrt{\frac{L}{g}}$$

(b)

$$\Delta t(l) = \frac{1}{\sqrt{g}}\int_0^{l}\frac{dx}{\sqrt{x}} = \frac{1}{\sqrt{g}}[2\sqrt{x}]_{x=0}^{x=l} = 2\sqrt{\frac{l}{g}}$$

$$2\sqrt{\frac{l}{g}} = \frac{1}{2}2\sqrt{\frac{L}{g}} \rightarrow l = \frac{L}{4}$$



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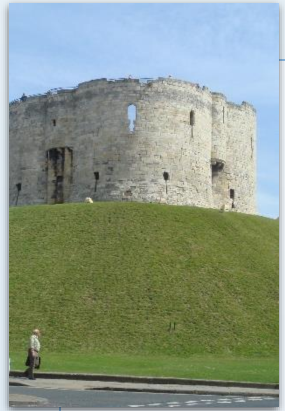

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