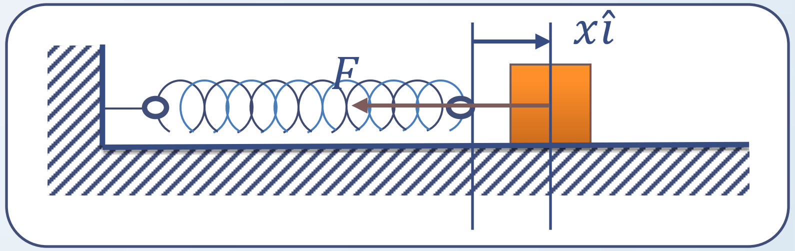
Chapter 15
Oscillation Motion-I

簡紋濱
國立交通大學 理學院 電子物理系

Outline

1. Motion of An Object Attached to A Spring
2. Simple Harmonic Motion
3. Energy of A Simple Harmonic Oscillator
4. Simple Harmonic Oscillator Versus Uniform Circular Motion
5. The Pendulum
6. Damped Oscillation
7. Forced Oscillation

1. MOTION OF AN OBJECT ATTACHED TO A SPRING



Hook's Law – restoring force

$$\vec{F} = -kx\hat{\imath}$$

Newton's 2nd Law, the block of mass m is reacting to the external, restoring force by the spring.

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{x}}{dt^2} = m\frac{d^2x}{dt^2}\hat{\imath} \rightarrow m\frac{d^2x}{dt^2} = -kx$$

When the block is acted by a linear restoring force, its motion follows a special oscillatory motion called simple harmonic motion (SHM).

The linear restoring force ($F = -kx$) gives the block in oscillation about the equilibrium position.

$$a = -\frac{kx}{m}$$

2. SIMPLE HARMONIC MOTION

The force equation and the differential equation of the SHM:

$$m\frac{d^2x}{dt^2} = -kx \rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Let $\boldsymbol{\omega^2 = k/m}$ $\qquad \frac{d^2x}{dt^2} + \omega^2 x = 0$

Guess an oscillatory function: $\boldsymbol{x(t) = A\cos(Bt + \phi)}$

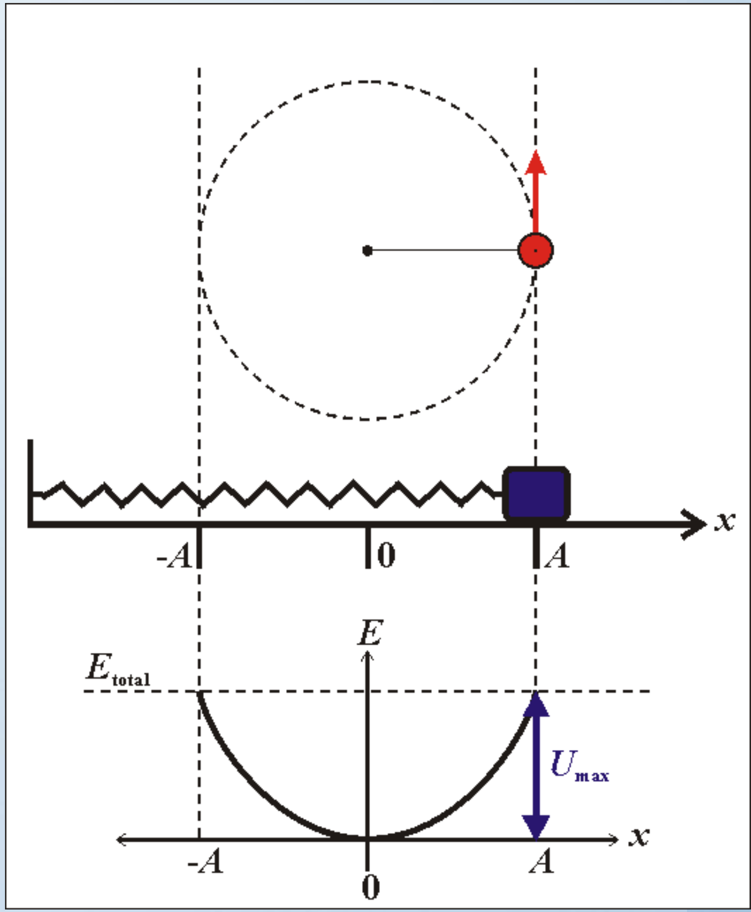
$$\frac{dx(t)}{dt} = -AB\sin(Bt + \phi) \qquad \frac{d^2x(t)}{dt^2} = -AB^2\cos(Bt + \phi)$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \rightarrow -AB^2\cos(Bt + \phi) + \omega^2 A\cos(Bt + \phi) = 0$$

$$A\cos(Bt + \phi)\,(\omega^2 - B^2) = 0 \rightarrow \mathrm{B} = \omega$$

$$x(t) = A\cos(\omega t + \phi)$$

angular speed: $\omega = \sqrt{k/m}$,
phase constant: ϕ,
period of time: $T = 2\pi\sqrt{m/k}$



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2. SIMPLE HARMONIC MOTION

The velocity and acceleration of the block

$$x(t) = A\cos(\omega t + \phi) \rightarrow v(t) = -\omega A\sin(\omega t + \phi)$$
$$\rightarrow a(t) = -\omega^2 A\cos(\omega t + \phi)$$

Giving the initial condition to evaluate the two parameters A, ϕ

1. At time $t = 0$, $x(0) = x_0$, $v(0) = 0$

$$A\cos\phi = x_0, -\omega A\sin\phi = 0$$

$$\phi = 0, A = x_0 \rightarrow x(t) = x_0\cos(\omega t), v(t) = -\omega x_0\sin(\omega t)$$

2. At time $t = 0$, $x(0) = 0$, $v(0) = v_0$

$$A\cos\phi = 0, -\omega A\sin\phi = v_0$$

$$\phi = \frac{\pi}{2}, A = -\frac{v_0}{\omega} \rightarrow x(t) = -\frac{v_0}{\omega}\cos\left(\omega t + \frac{\pi}{2}\right)$$

$$x(t) = \frac{v_0}{\omega}\cos\left(\frac{\pi}{2} - \omega t\right) = \frac{v_0}{\omega}\sin(\omega t) \qquad v(t) = v_0\cos(\omega t)$$

2. SIMPLE HARMONIC MOTION

Example: A block with a mass of 200 g is connected to a light horizontal spring of force constant 5.0 N/m and is free to oscillate on a horizontal, frictionless surface.
(a) If the block is displaced 5.0 cm from equilibrium and released from rest, find the period of its motion. (b) Determine the maximum speed and the maximum acceleration of the block.

$$m = 0.20\ kg, k = 5.0\ N/m$$

$$\omega^2 = \frac{k}{m} = 25 \rightarrow \omega = 5.0\ rad/s$$

(a) $T = \dfrac{2\pi}{\omega} = 1.3\ s$ $\quad t = 0, x(t = 0) = x_0 = 5.0\ cm = 0.050\ m$

(b) $x(t) = x_0 \cos(\omega t) = 0.050 \cos(5.0t)$

$v(t) = -0.25 \sin(5.0t) \quad v_{max} = 0.25\ m/s$

$a(t) = -1.3 \cos(5.0t) \quad a_{max} = 1.3\ m/s^2$

2. SIMPLE HARMONIC MOTION

Example: Suppose that the initial position x_i and the initial velocity v_i of a harmonic oscillator of known angular frequency ω are given: that is $x(0) = x_i, v(0) = v_i$. Find general expression for the amplitude and the phase constant in terms of these initial parameters.

typical positional function: $x(t) = A\cos(\omega t + \phi)$

$v(t) = -\omega A\sin(\omega t + \phi)$

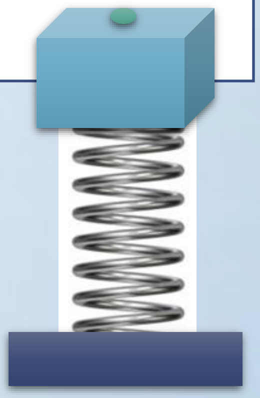
the initial conditions give: $x_i = A\cos\phi\,, v_i = -\omega A\sin\phi$

$$x_i^2 + \left(\frac{v_i}{\omega}\right)^2 = A^2 \rightarrow A = \sqrt{x_i^2 + \left(\frac{v_i}{\omega}\right)^2}$$

$$\frac{v_i}{x_i} = -\omega\tan\phi \rightarrow \phi = \tan^{-1}\left(-\frac{v_i}{\omega x_i}\right)$$

2. SIMPLE HARMONIC MOTION

Example: A block attached to a spring oscillates vertically with a frequency of 4.0 Hz and an amplitude of 7.0 cm. A tiny bead is placed on top of the oscillating block just as it reaches its lowest point. Assume that the bead's mass is so small that its effect on the motion of the block is negligible. At what distance from the block's equilibrium position does the bead lose contact with the block?



$\omega = 2\pi f = 4 \times 2\pi = 25\, rad/s$

the gravitational acceleration g helps to place the bead on top of the block, if the restoring force gives an downward acceleration larger than g, the bead starts to leave the block

the acceleration of the block: $ma = kx \rightarrow a = \frac{k}{m}x = \omega^2 x$

$a \geq g \rightarrow \omega^2 x \geq g \rightarrow x \geq \frac{g}{\omega^2} \rightarrow x \geq 0.016\, m$

At a height of 1.6 cm above the equilibrium position of the spring-block system

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3. ENERGY OF A SIMPLE HARMONIC OSCILLATOR

Start from a positional function $x(t) = A\cos(\omega t + \phi)$

$$v(t) = -\omega A \sin(\omega t + \phi)$$

The kinetic energy as a function of time:

$$K = \frac{1}{2} m v(t)^2 = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi)$$

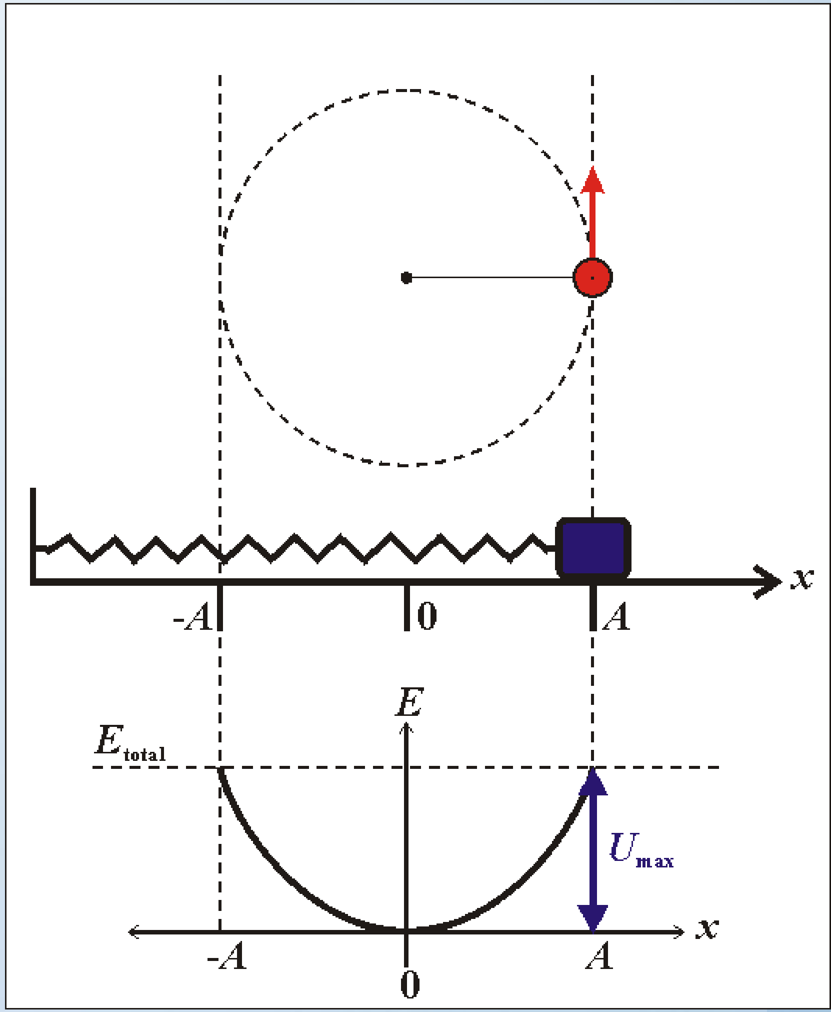
The potential energy as a function of time:

$$U = \frac{1}{2} k x(t)^2 = \frac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

The net mechanical energy:

$$E = K + U = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi) + \frac{1}{2} k A^2 \cos^2(\omega t + \phi) = \frac{1}{2} k A^2$$

$$\frac{1}{2} m v^2 + \frac{1}{2} k x^2 = \frac{1}{2} k A^2 \rightarrow v^2 = \frac{k}{m}(A^2 - x^2) \rightarrow v = \pm\omega\sqrt{A^2 - x^2}$$



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3. ENERGY OF A SIMPLE HARMONIC OSCILLATOR

Example: A 0.50 kg object connected to a massless spring of force constant 20 N/m oscillates on a horizontal, frictionless track. (a) Calculate the total energy of the system and the maximum velocity of the object if the amplitude of the motion is 3.0 cm. (b) What is the velocity of the object when the position is equal to 2.0 cm?

$A = 3.0\ cm = 0.030\ m$

(a)

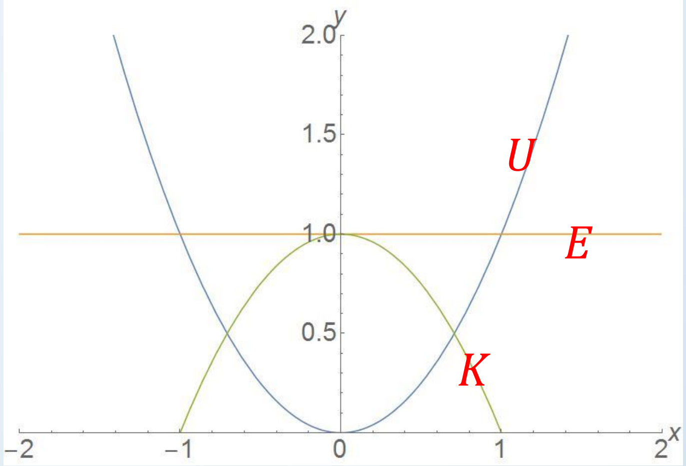
$$E = \frac{1}{2}kA^2 = \frac{1}{2}(20)(0.030)^2 = 0.0090\ J$$

$$\frac{1}{2}mv_{max}^2 = E \rightarrow v_{max} = \sqrt{2E/m} = \sqrt{2 \times 0.0090/0.50} = 0.19\ m/s$$

(b)

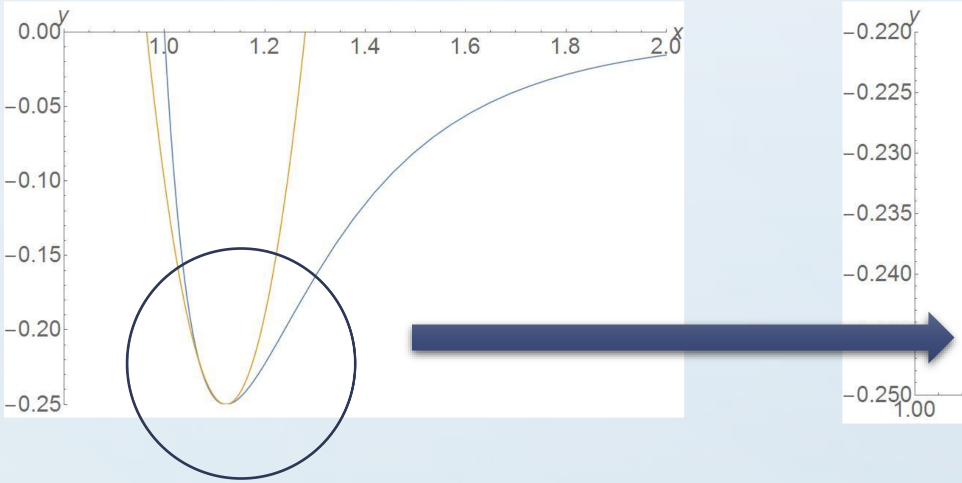
$$v = \pm\sqrt{k/m}\sqrt{(A^2 - x^2)} = \pm 6.3 \times 0.022 = \pm 0.14\ m/s$$

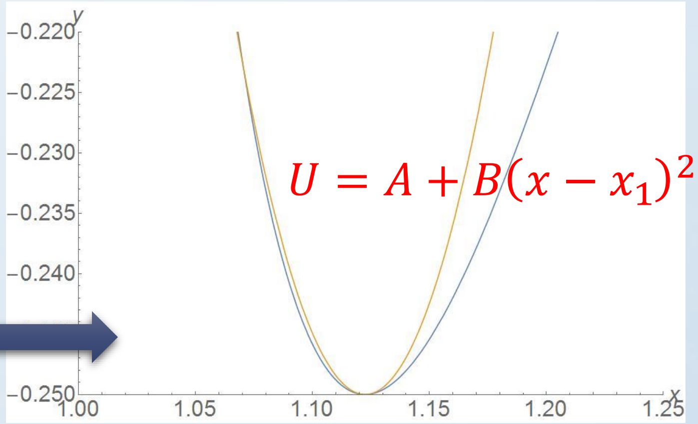
3. ENERGY OF A SIMPLE HARMONIC OSCILLATOR



The potential function, the kinetic energy, and the total energy

General motion near equilibrium, for example, the Lenard-Jones potential energy





4. SIMPLE HARMONIC OSCILLATOR VERSUS UNIFORM CIRCULAR MOTION

Polar coordinate

$\theta = \omega t + \phi$

$\vec{r}(t) = r\cos(\omega t + \phi)\,\hat{\imath} + r\sin(\omega t + \phi)\,\hat{\jmath}$

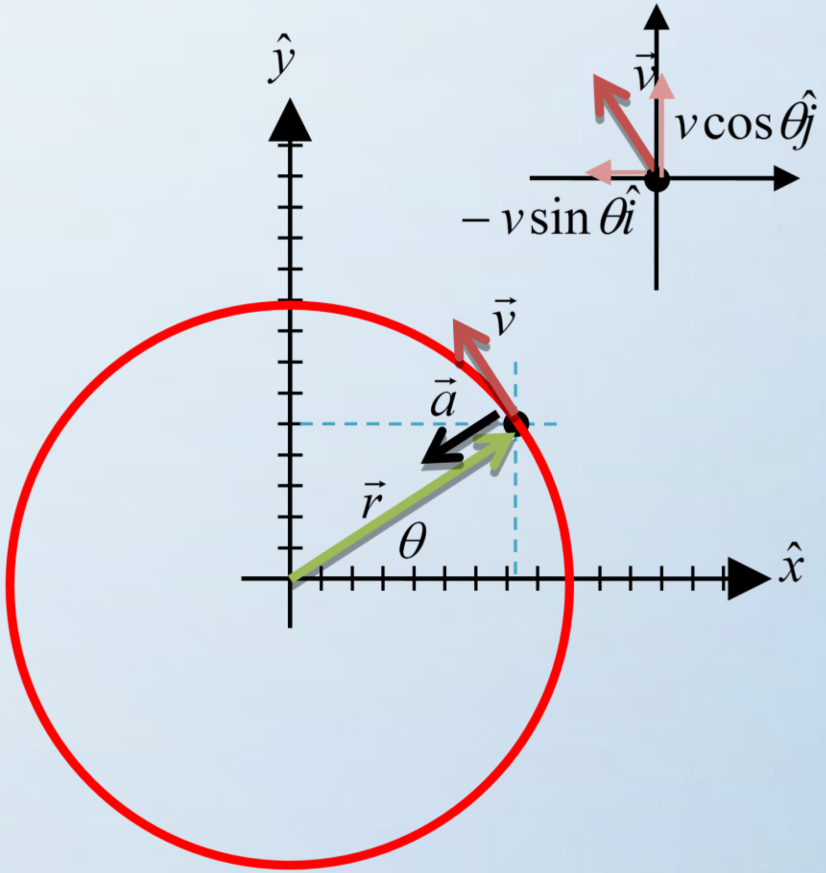
$\vec{v}(t) = -\omega r\sin(\omega t + \phi)\,\hat{\imath} + \omega r\cos(\omega t + \phi)\,\hat{\jmath}$

$\vec{a}(t) = -\omega^2 r\cos(\omega t + \phi)\,\hat{\imath} - \omega^2 r\sin(\omega t + \phi)\,\hat{\jmath}$

$x(t) = r\cos(\omega t + \phi)$

$v_x(t) = -\omega r\sin(\omega t + \phi)$

$a_x(t) = -\omega^2 r\cos(\omega t + \phi)$



5. THE PENDULUM

Pendulum

restoring force: $F = -mg\sin\theta$

Newton's law: $F = ma = m\frac{d^2s}{dt^2} = mL\frac{d^2\theta}{dt^2}$

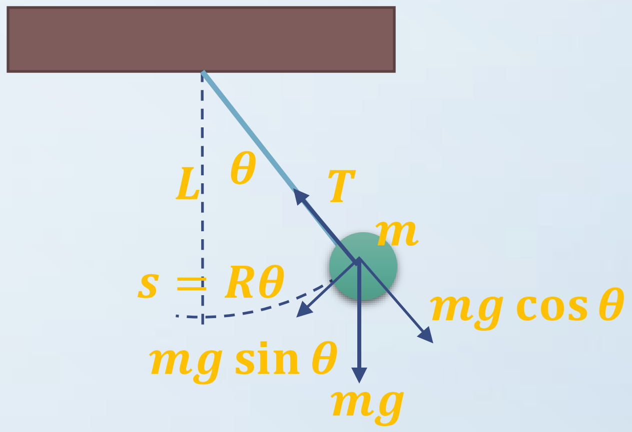
The force EQ (differential EQ):

$$mL\frac{d^2\theta}{dt^2} = -mg\sin\theta \rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

If the angle θ is small enough, the differential equation is

$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$ compare it with $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$

$$\omega = \sqrt{\frac{g}{L}}, T = 2\pi\sqrt{\frac{L}{g}}$$



5. THE PENDULUM

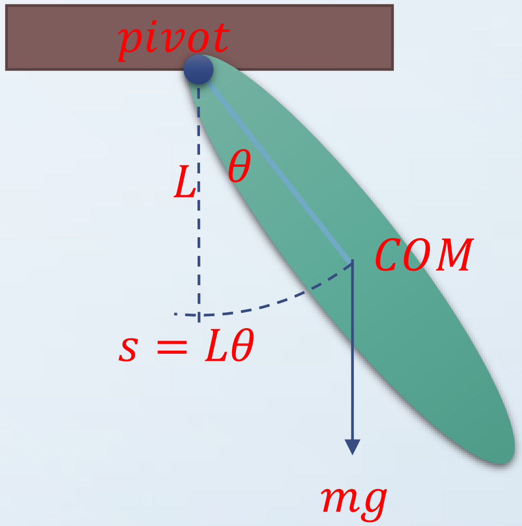
Physical Pendulum

restoring torque: $\tau = -Lmg\sin\theta$

Newton's law: $\tau = I\alpha = I\frac{d^2\theta}{dt^2}$

The torque EQ (differential EQ):

$$I\frac{d^2\theta}{dt^2} = -Lmg\sin\theta \rightarrow \frac{d^2\theta}{dt^2} + \frac{Lmg}{I}\sin\theta = 0$$



If the angle θ is small enough, the differential equation is

$\frac{d^2\theta}{dt^2} + \frac{Lmg}{I}\theta = 0$ compare it with $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$

$$\omega = \sqrt{\frac{Lmg}{I}}, T = 2\pi\sqrt{\frac{I}{Lmg}}$$

5. THE PENDULUM

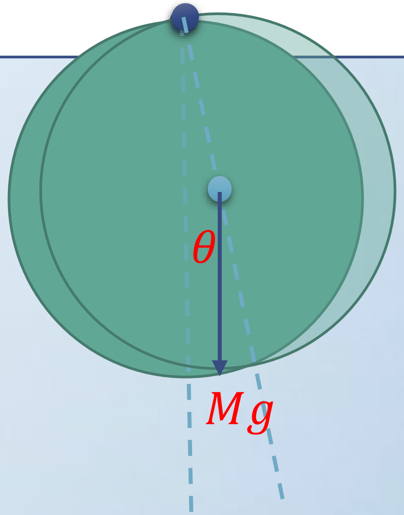
Example: A man enters a tall tower and he wants to measure its height. He puts a long pendulum extending from the ceiling almost to the floor and he measures a period of 12 s. How tall is the tower?

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

$$T = 2\pi\sqrt{\frac{L}{g}} = 12 \rightarrow L = g(12/2\pi)^2 = 36\ m$$

5. THE PENDULUM

Example: A circular sign of mass M and radius R is hung on a nail from a small loop located at one edge. After it is placed on the nail, the sign oscillates in a vertical plane. Find the period of oscillation if the amplitude is small.

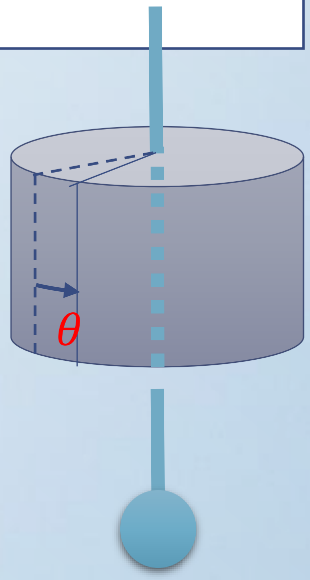


$$I_p = I_{COM} + MR^2 = \frac{3}{2}MR^2$$

$$T = 2\pi\sqrt{\frac{I}{Rmg}} = 2\pi\sqrt{\frac{3MR^2/2}{RMg}} = 2\pi\sqrt{\frac{3R}{2g}}$$

5. THE PENDULUM

Example: A rigid object suspended by a wire attached at the top to a fixed support. Assume that the inertia of momentum is I. When the object is twisted through some angle θ, the twisted wire exerts on the object a restoring torque that is proportional to the angular position. That is, $\tau = -\kappa\theta$, where κ is called the torsion constant of the support wire. Find the period of oscillation.



$$\tau = I\alpha = -\kappa\theta$$

$$I\frac{d^2\theta}{dt^2} + \kappa\theta = 0$$

$$\omega = \sqrt{\frac{\kappa}{I}}$$

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$

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6. DAMPED OSCILLATION

The external force:

$$F = -kx - bv$$

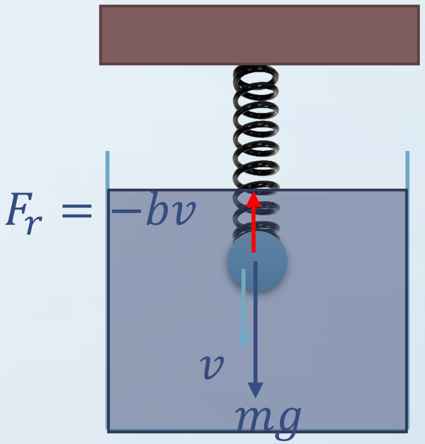
Newton's 2nd Law:

$$F = ma$$

The force equation (differential equation):

$$ma = -kx - bv$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$



Guess solution: $x(t) = Ae^{at}$

$$ma^2Ae^{at} + baAe^{at} + kAe^{at} = 0 \rightarrow (ma^2 + ba + k)Ae^{at} = 0$$

$$a = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \qquad x(t) = A_1e^{\frac{-b+\sqrt{b^2-4mk}}{2m}t} + A_2e^{\frac{-b-\sqrt{b^2-4mk}}{2m}t}$$

6. DAMPED OSCILLATION

Overdamped oscillation: $b^2 - 4mk > 0$

$$x(t) = A_1 e^{-\frac{b}{2m}t + \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}t} + A_2 e^{-\frac{b}{2m}t - \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}t}$$

Critically damped oscillation: $b^2 - 4mk = 0$

$$x(t) = A_1 e^{-\frac{b}{2m}t}$$

Underdamped oscillation: $b^2 - 4mk < 0$

$$\sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} = i\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = i\omega$$

$$x(t) = e^{-\frac{b}{2m}t}\left(A_1 e^{+i\omega t} + A_2 e^{-i\omega t}\right)$$

$$x(t) = e^{-\frac{b}{2m}t} A \sin(\omega t + \phi)$$

7. FORCED OSCILLATION

The external force:

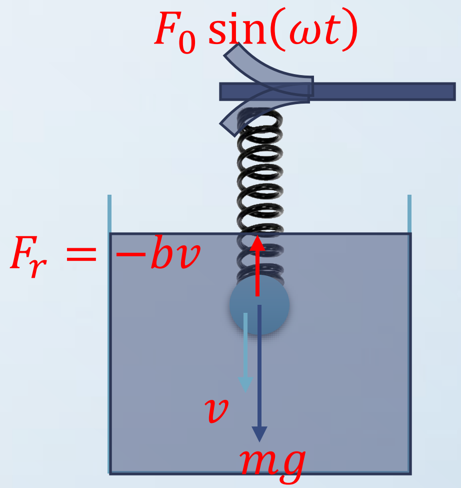
$$F = -kx - bv + F_0 \sin(\omega t)$$

The force equation (differential equation):

$$ma = -kx - bv + F_0 \sin(\omega t)$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \sin(\omega t)$$

Guess solution: $\boldsymbol{x(t) = A \sin(\omega t + \phi)}$



$$(k - m\omega^2)A \sin(\omega t + \phi) + \omega b A \cos(\omega t + \phi) = F_0 \sin(\omega t)$$

$$(k - m\omega^2)A \cos\phi \sin(\omega t) + (k - m\omega^2)A \sin\phi \cos(\omega t)$$
$$+\omega b A \cos\phi \cos(\omega t) - \omega b A \sin\phi \sin(\omega t) = F_0 \sin(\omega t)$$

$$(k - m\omega^2) \sin\phi = -\omega b \cos\phi \rightarrow \left(\omega^2 - \frac{k}{m}\right) \sin\phi = \omega \frac{b}{m} \cos\phi$$

7. FORCED OSCILLATION

$$\tan\phi = \frac{\omega\frac{b}{m}}{\left(\omega^2 - \frac{k}{m}\right)} = \frac{-\omega\frac{b}{m}}{\left(\frac{k}{m} - \omega^2\right)}$$

$$A\big((k - m\omega^2)\cos\phi\sin(\omega t) - \omega b\sin\phi\sin(\omega t)\big) = F_0\sin(\omega t)$$

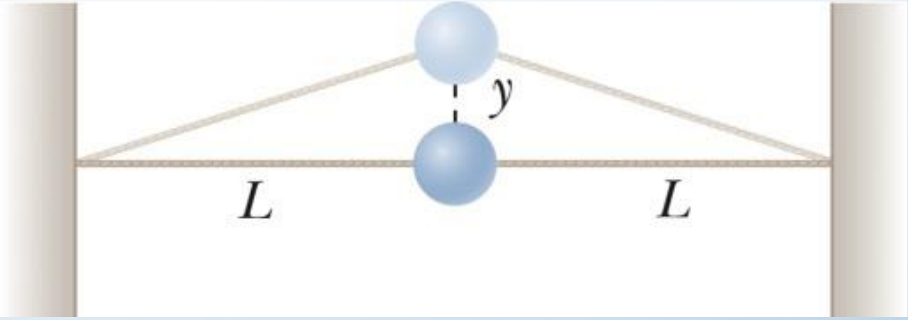
$$\sqrt{\left(\frac{k}{m} - \omega^2\right)^2 + \left(\frac{b\omega}{m}\right)^2}\,A\left(\frac{\frac{k}{m} - \omega^2}{\sqrt{\left(\frac{k}{m} - \omega^2\right)^2 + \left(\frac{b\omega}{m}\right)^2}}\cos\phi + \frac{-\omega\frac{b}{m}}{\sqrt{\left(\frac{k}{m} - \omega^2\right)^2 + \left(\frac{b\omega}{m}\right)^2}}\sin\phi\right) = \frac{F_0}{m}$$

$$\sqrt{\left(\frac{k}{m} - \omega^2\right)^2 + \left(\frac{b\omega}{m}\right)^2}\,A(\cos^2\phi + \sin^2\phi) = \frac{F_0}{m}$$

$$A = \frac{F_0/m}{\sqrt{\left(\frac{k}{m} - \omega^2\right)^2 + \left(\frac{b\omega}{m}\right)^2}}$$

EXERCISE

A ball of mass m is connected to two rubber bands of length L, each under tension T. The ball is displaced by a small distance y perpendicular to the length of the rubber bands. Assuming that the tension does not change, please calculate the restoring force and the angular frequency for this system in simple harmonic motion.



$$F_y = 2T\sin(\theta) = 2T\frac{y}{\sqrt{L^2+y^2}}$$

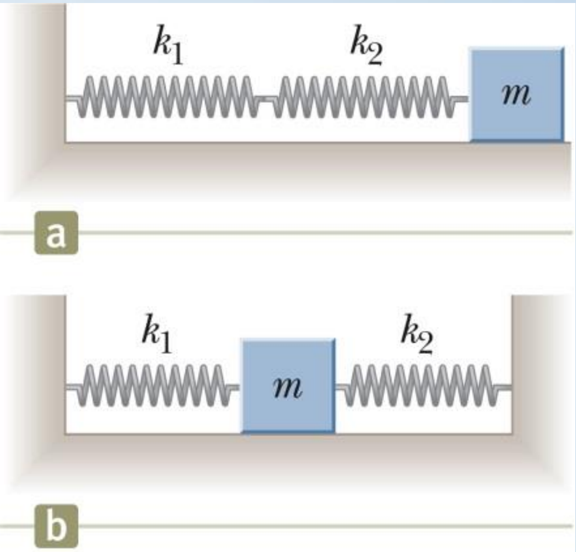
The force equation gives $F = ma = -2T\frac{y}{\sqrt{L^2+y^2}}$

$$a = \frac{d^2y}{dt^2} \rightarrow m\frac{d^2y}{dt^2} + 2T\frac{y}{\sqrt{L^2+y^2}} = 0$$

$$y \ll L \rightarrow m\frac{d^2y}{dt^2} + \frac{2T}{L}y = 0 \rightarrow \omega^2 = \frac{2T}{mL} \rightarrow \omega = \sqrt{\frac{2T}{mL}}$$

EXERCISE

A block of mass m is connected to two springs of force constants k_1 and k_2 in two ways as shown in the figure. In both cases, the block moves on the frictionless table after it is displaced from equilibrium and released. Show that in the two cases the block exhibits simple harmonic oscillation with periods (a) $T = 2\pi\sqrt{m(k_1 + k_2)/k_1 k_2}$ and (b) $T = 2\pi\sqrt{m/(k_1 + k_2)}$.



a. The same F, $F = k_1 x_1 = k_2 x_2 = k_{eff} x_t$

Total elongation $x_t = x_1 + x_2$

$$F = k_{eff} x_t = k_{eff}(x_1 + x_2)$$

$$F = k_{eff}\left(\frac{F}{k_1} + \frac{F}{k_2}\right) \rightarrow k_{eff} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$$

$$k_{eff} = \frac{k_1 k_2}{k_1 + k_2}$$

$$T = 2\pi\sqrt{\frac{m}{k_{eff}}} = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$

EXERCISE

A block of mass m is connected to two springs of force constants k_1 and k_2 in two ways as shown in the figure. In both cases, the block moves on the frictionless table after it is displaced from equilibrium and released. Show that in the two cases the block exhibits simple harmonic oscillation with periods (a) $T = 2\pi\sqrt{m(k_1 + k_2)/k_1 k_2}$ and (b) $T = 2\pi\sqrt{m/(k_1 + k_2)}$.

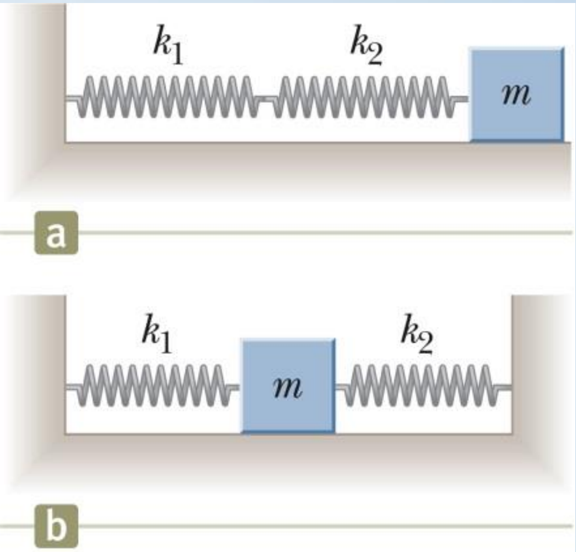
b. The same elongation, $x_1 = x_2 = x$

Total force $F_t = F_1 + F_2$

$$F = F_1 + F_2 = k_{eff}x$$

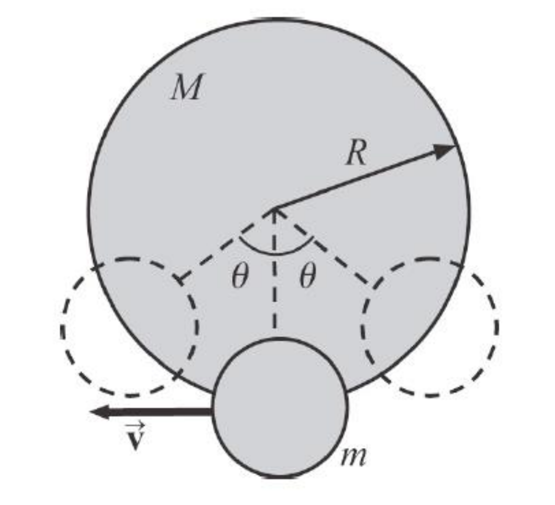
$$k_1 x_1 + k_2 x_2 = k_{eff}x \rightarrow k_{eff} = k_1 + k_2$$

$$T = 2\pi\sqrt{\frac{m}{k_{eff}}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$



EXERCISE

A smaller disk of radius r and mass m is attached rigidly to the face of a second larger disk of radius R and mass M. The center of the small disk is located at the edge of the large disk. The large disk is mounted at its center on a frictionless axle. The assembly is rotated through a small angle θ from its equilibrium position and released. (a) Please calculate the speed of the center of the small disk as it passes through the equilibrium position. (b) Please calculate the period of the motion.



Calculate the moment of inertia about the axle:

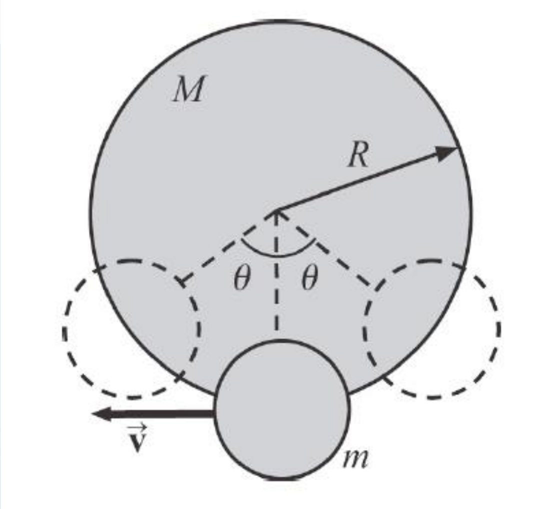
$$I_{M,axle} = \frac{MR^2}{2}$$

$$I_{m,CM} = \frac{mr^2}{2} \rightarrow I_{m,axle} = \frac{mr^2}{2} + mR^2$$

$$I_{total} = \frac{MR^2}{2} + \frac{mr^2}{2} + mR^2$$

EXERCISE

A smaller disk of radius r and mass m is attached rigidly to the face of a second larger disk of radius R and mass M. The center of the small disk is located at the edge of the large disk. The large disk is mounted at its center on a frictionless axle. The assembly is rotated through a small angle θ from its equilibrium position and released. (a) Please calculate the speed of the center of the small disk as it passes through the equilibrium position. (b) Please calculate the period of the motion.



Change the potential energy to rotational energy:

$$mg(1-\cos(\theta))R = \frac{1}{2} I_{total}\omega^2 = \frac{1}{2} I_{total}\left(\frac{v}{R}\right)^2$$

$$v^2 = R^2 \frac{2mgR(1-\cos(\theta))}{\frac{MR^2}{2} + \frac{mr^2}{2} + mR^2}$$

$$v = 2\sqrt{\frac{gR(1-\cos(\theta))}{\frac{M}{m} + \frac{r^2}{R^2} + 2}}$$

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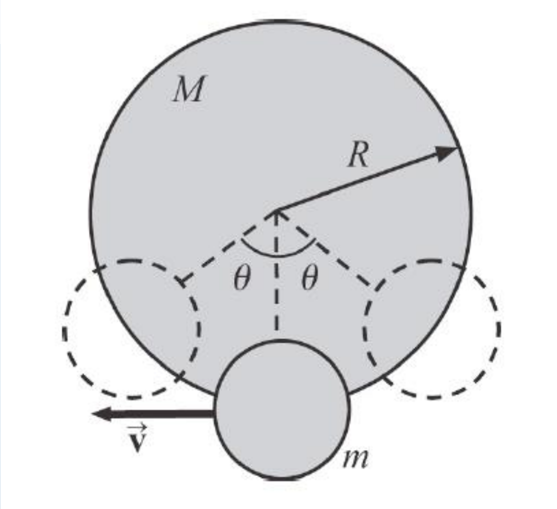

The torque equation is:

$$I_{total}\alpha = -R \cdot mg \cdot \sin(\theta)$$

$$\left(\frac{MR^2}{2} + \frac{mr^2}{2} + mR^2\right)\frac{d^2\theta}{dt^2} + mgR\sin(\theta) = 0$$

Small angle approximation gives:

$$\left(\frac{MR^2}{2} + \frac{mr^2}{2} + mR^2\right)\frac{d^2\theta}{dt^2} + mgR\theta = 0$$

$$\omega = \sqrt{\frac{mgR}{\frac{MR^2}{2} + \frac{mr^2}{2} + mR^2}}$$

ACKNOWLEDGEMENT

國立交通大學理學院
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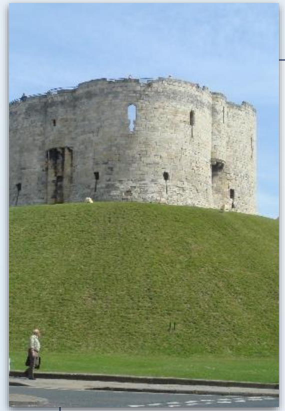

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