Chapter 32 Alternative-Current Circuits

Physics II – Part II

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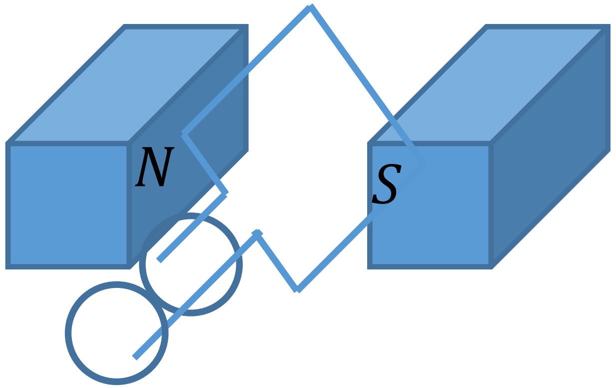
Alternative Current Voltage Source

AC Sources



The power generator utilizes a changing magnetic flux where mechanical energy is converted to a variation of magnetic flux variation.

$$\Phi_B = \vec{B} \cdot \vec{A} = BA\cos(\theta)$$

Note that the flux variation could be activated using a changing magnetic field, a changing area, or a changing angle between the magnetic field and the areal normal vector.

The mechanical energy causing a rotational motion may be the easiest way to vary a magnetic flux.

$$\theta = \omega t$$

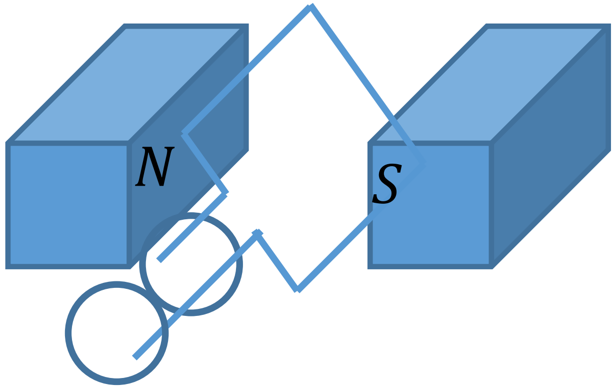
$$\Phi_B = \vec{B} \cdot \vec{A} = BA\cos(\omega t)$$

Consequently, this is the most common way for power plant and it results in the alternative voltage on the outlet plug in your house.

Ref:

Alternative Current Voltage Source

The Average of AC Voltage Sources



The power plug supplies a power of AC voltage.

$$\varepsilon = V = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(BA\cos(\omega t))$$

$$\varepsilon = \omega BA\sin(\omega t)$$

It comes a general description of the AC voltage as:

$$\theta = \omega t + \varphi \rightarrow \varepsilon = \omega BA\sin(\omega t + \varphi) = V_0\sin(\omega t + \varphi)$$

The calculation average voltage and the root-mean-square voltage are:

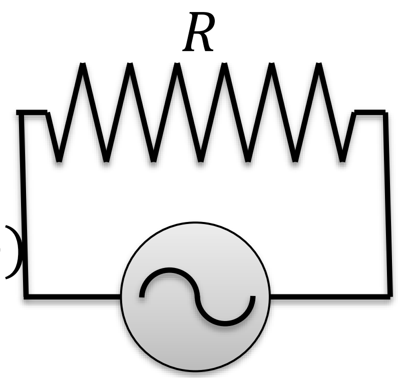
$$V_{avg} = \frac{1}{T}\int_0^T V_0\sin(\omega t + \varphi)\,dt = 0$$

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T V_0^2\sin^2(\omega t + \varphi)\,dt} = \sqrt{\frac{1}{T}V_0^2\frac{T}{2}} = \frac{V_0}{\sqrt{2}}$$

Ref:

Resistors in an AC Circuit

Resistors in AC Circuits



Giving an AC voltage across a resistor, what is the current in the resistor?

The general form of an AC voltage is $V_0 \sin(\omega t + \varphi)$

The current through the resistor is evaluated

$$I(t) = \frac{V(t)}{R} = \left(\frac{V_0}{R}\right) \sin(\omega t + \varphi) = I_0 \sin(\omega t + \varphi)$$

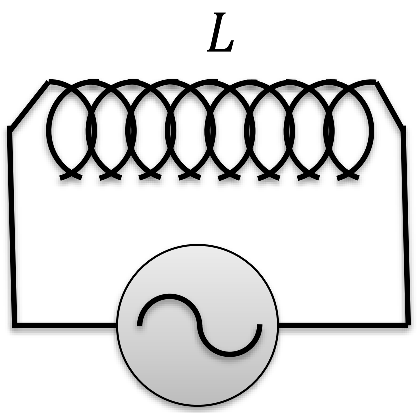
Using the Ohm's law in AC voltage, it gives

$$V_0 = I_0 R$$

Note that the current and the voltage of the resistor have the same phase of $+\varphi$.

Ref:

The Inductors in AC Circuits



Assume the AC voltage across the inductor of

$V_0 \sin(\omega t + \varphi)$

The current in the inductor is evaluated:

$$V = L\frac{dI}{dt} \rightarrow I = \frac{V_0}{L}\int \sin(\omega t + \varphi)\, dt = -\frac{V_0}{\omega L}\cos(\omega t + \varphi)$$

$$I = -\frac{V_0}{\omega L}\sin\left(\frac{\pi}{2} - \omega t - \varphi\right) = \frac{V_0}{\omega L}\sin\left(\omega t + \varphi - \frac{\pi}{2}\right)$$

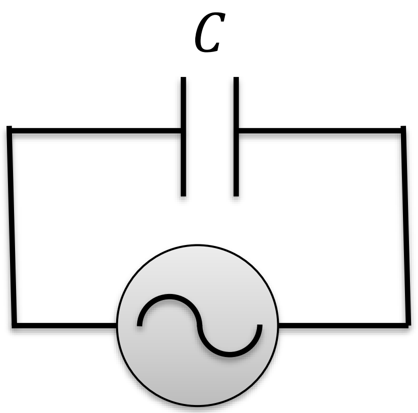
The impedance of the inductor is:

$$I_0 = \frac{V_0}{\omega L} \rightarrow X_L = \omega L$$

The current flowing through the inductor has a negative phase shift of $\frac{\pi}{2}$ compared with the voltage across the inductor.

Ref:

The Capacitors in AC Circuits



Assume the AC voltage across the capacitor of

$V_0 \sin(\omega t + \varphi)$

The current in the capacitor is estimated:

$$C = \frac{Q}{V} \rightarrow I = C\frac{dV}{dt} = \omega C V_0 \cos(\omega t + \varphi)$$

$$I = \omega C V_0 \sin\left(\frac{\pi}{2} - \omega t - \varphi\right) = \omega C V_0 \sin\left(\omega t + \varphi + \frac{\pi}{2}\right)$$

The impedance of the capacitor is:

$$I_0 = \omega C V_0 \rightarrow X_C = 1/\omega C$$

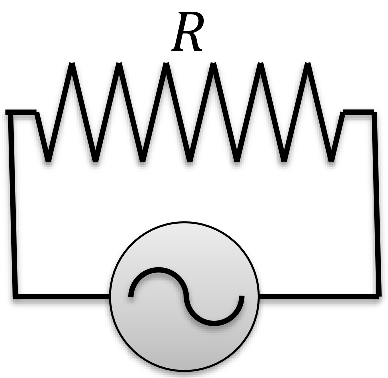
The current flowing through the inductor has a positive phase shift of $\frac{\pi}{2}$ compared with the voltage across the capacitor.

Ref:

Resistors in an AC Circuit

The Phasor Concept of The Resistor in AC Circuits

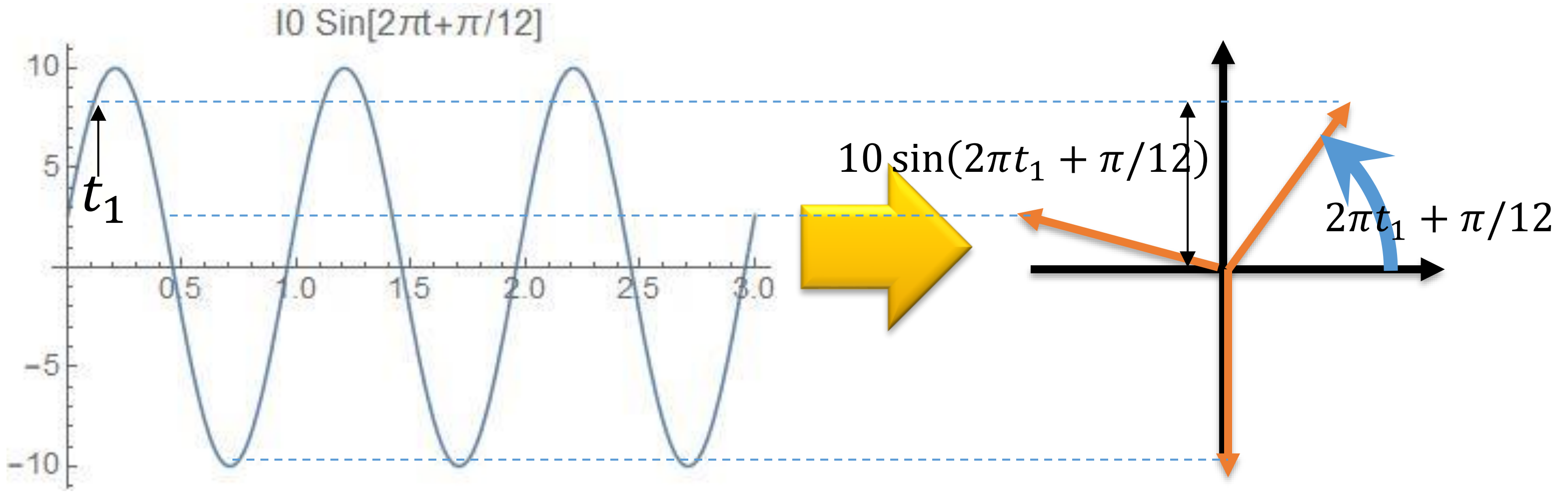
To deploy a phasor scenario, you better know that the circuits are connected in either serial or parallel.



For a parallel connection, the voltage source is fixed thus we look into the current.

$$I(t) = \left(\frac{V_0}{R}\right)\sin(\omega t + \varphi)$$

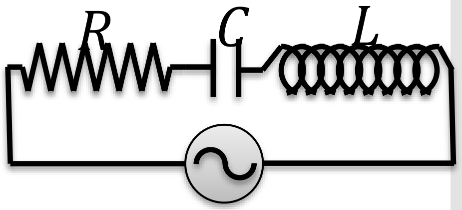
We draw the current in a two-dimensional map to work on the phasor calculation.



Ref:

Series RLC Circuit

Find the current of the RLC circuit connected in series and driven by an AC voltage of $V_0 \sin(\omega t)$.



The voltage source is $V_0 \sin(\omega t)$.

The circuit is connected in series thus giving the same current in the electronic components.

Through the capacitor, the voltage slower than the current for $\pi/2$.

Through the inductor, the voltage is faster than the current for $\pi/2$.

If the voltage across the capacitor is larger than that across the inductor, the current will be $I = I_0 \sin(\omega t + \delta)$.

If the voltage across the inductor is larger than that across the capacitor, the current will be $I = I_0 \sin(\omega t - \delta)$.

Ref:

The RLC Series Circuit

Series RLC Circuit

Find the current of the RLC circuit connected in series and driven by an AC voltage of $V_0 \sin(\omega t)$.

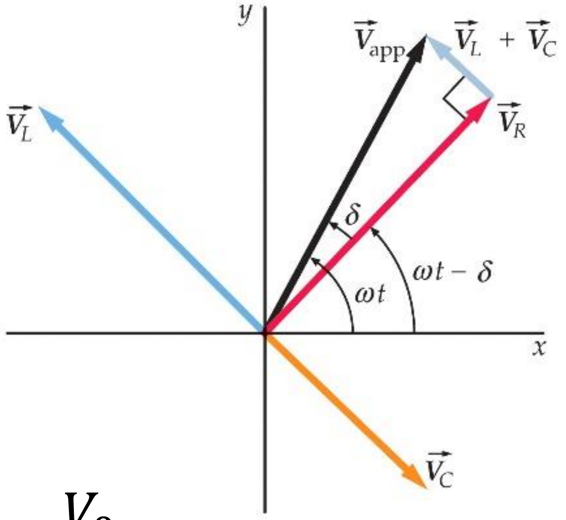
The voltage source is $V_0 \sin(\omega t)$.

Assume $V_L > V_C$ & $I = I_0 \sin(\omega t - \delta)$.

$$V_R = I_0 R \sin(\omega t - \delta)$$

$$V_L = I_0 \omega L \sin\left(\omega t - \delta + \frac{\pi}{2}\right)$$

$$V_C = \frac{I_0}{\omega C} \sin\left(\omega t - \delta - \frac{\pi}{2}\right)$$



$$V_0 = \sqrt{I_0^2 R^2 + I_0^2 \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$I_0 = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$\tan(\delta) = \frac{\omega L - 1/\omega C}{R}$$

Ref:

Power Consumption in an AC Circuit

Power Delivered by Voltage Source

Assume that the AC voltage across the resistor of a resistance R is $V(t) = V_0 \sin(\omega t)$, the current in the resistor will be $I(t) = (V_0/R)\sin(\omega t) = I_0 \sin(\omega t)$.

$$P(t) = I(t)V(t) = \frac{V_0^2}{R}\sin^2(\omega t)$$

$$P_{avg} = \frac{1}{T}\int_0^T \frac{V_0^2}{R}\sin^2(\omega t)\,dt = \frac{V_0^2}{2R}$$

Since $V_{rms} = V_0/\sqrt{2}$, $P_{avg} = \frac{V_{rms}^2}{R} = I_{rms}^2 R$

The average power delivered by a generator of an AC voltage source is

$$P(t) = V_0 \sin(\omega t)\, I_0 \sin(\omega t) \rightarrow P_{avg} = \frac{1}{2} I_0 V_0 = I_{rms} V_{rms}$$

Ref:

Power Consumption in a series RLC Circuit

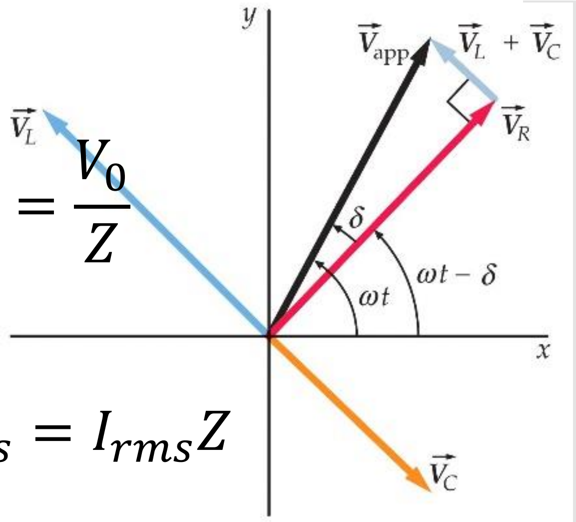
Power Delivered by Voltage Source

The voltage source is $V_0 \sin(\omega t)$.

$$V_R = I_0 R \sin(\omega t - \delta) \quad I_0 = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V_0}{Z}$$

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$V_0 = I_0 Z \rightarrow V_{rms} = I_{rms} Z$$



$$\tan(\delta) = \frac{\omega L - 1/\omega C}{R} \rightarrow \cos(\delta) = \frac{R}{Z}$$

The power delivered from **the voltage source** is

$$P(t) = V_0 \sin(\omega t)\, I_0 \sin(\omega t - \delta)$$

$$P(t) = I_0 V_0 (\sin^2(\omega t)\cos(\delta) - \sin(\omega t)\cos(\omega t)\sin(\delta))$$

$$P_{avg} = \frac{I_0 V_0}{2}\cos(\delta) = I_{rms} V_{rms} \frac{R}{Z} = I_{rms}^2 R = I_{rms}^2 Z$$

Ref:

Power Consumption in a series RLC Circuit

Power Dissipated on Resistor

The voltage source is $V_0 \sin(\omega t)$ and the current is $I_0 \sin(\omega t - \delta)$.

$$V_0 = I_0 Z \rightarrow V_{rms} = I_{rms} Z \qquad Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

The power dissipated on the resistor is

$$P(t) = I_0^2 \sin^2(\omega t - \delta)\, R \rightarrow P_{avg} = \frac{I_0^2 R}{2} = I_{rms}^2 R = \left(\frac{V_{rms}}{Z}\right)^2 R$$

$$P_{avg} = \frac{V_0^2 R}{2Z^2} = \frac{1}{2} \frac{V_0^2 R}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

The dissipated power on the resistor has a maximum value at $\omega L - \frac{1}{\omega C} = 0$. The frequency is named the resonance frequency $\omega_r = 1/\sqrt{LC}$.

The full width at half maximum is calculated by $\omega L - \frac{1}{\omega C} = R$.

Ref:

Power Consumption in a series RLC Circuit

Resonance in The RLC Circuit

The full width at half maximum occurs at $\omega L - \frac{1}{\omega C} = R$.

Let $\omega = \omega_0 + \Delta\omega$ and $\omega_0 \gg \Delta\omega$.

$$\omega L - \frac{1}{\omega C} = R \rightarrow \frac{\omega^2 LC - 1}{\omega_0 C} = R$$

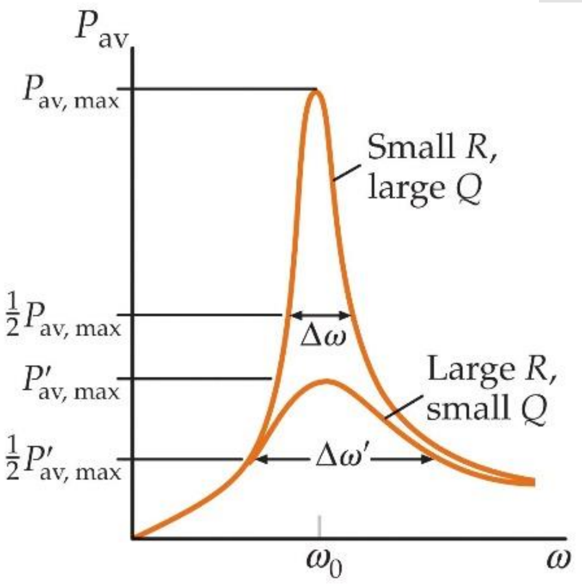
$$\omega^2 \cong \omega_0^2 + 2\omega_0 \Delta\omega$$

$$\omega^2 LC - 1 = R\omega_0 C \rightarrow \omega^2 = \omega_0^2 + \omega_0 \frac{R}{L}$$

$$\rightarrow 2\Delta\omega = \frac{R}{L}$$

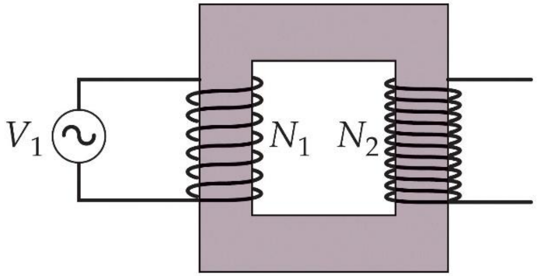
The full width at half maximum is R/L.

The quality factor is defined by $Q = \frac{\omega_0}{2\Delta\omega} = \frac{\omega_0 L}{R}$.



Ref:

Transformer



Assume the magnetic flux of a single loop to be ϕ.

There are N_1 turns on the side of the voltage source.

$$V_1 + N_1 \frac{d\phi}{dt} = 0 \rightarrow \frac{d\phi}{dt} = -\frac{V_1}{N_1}$$

There are N_2 turns on the side of the transformed voltage.

$$V_2 + N_2 \frac{d\phi}{dt} = 0 \rightarrow V_2 = -N_2 \frac{d\phi}{dt} = \frac{N_2}{N_1} V_1$$

The energy must be conserved thus $I_1 V_1 = I_2 V_2$.

Ref:

Root-Mean Square Values

Examples – The Average Value of AC Voltage

Find the average voltage and root-mean square voltage for the periodic saw tooth waveform of $V(t) = V_0 \frac{t}{T} (0 < t < T)$.

$$V_{avg} = \frac{1}{T}\int_0^T V_0 \frac{t}{T} dt = V_0 \frac{1}{T^2}\frac{T^2}{2} = \frac{V_0}{2}$$

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T V_0^2 \frac{t^2}{T^2} dt} = \sqrt{\frac{1}{T} V_0^2 \frac{1}{T^2}\frac{T^3}{3}} = \frac{V_0}{\sqrt{3}}$$

Ref:

Examples – The Average Value of AC Voltage

Find the average voltage and root-mean square voltage for the periodic saw tooth waveform of $V(t) = V_0 \frac{t^2}{T^2} (0 < t < T)$.

$$V_{avg} = \frac{1}{T}\int_0^T V_0 \frac{t^2}{T^2} dt = V_0 \frac{1}{T^3}\frac{T^3}{3} = \frac{V_0}{3}$$

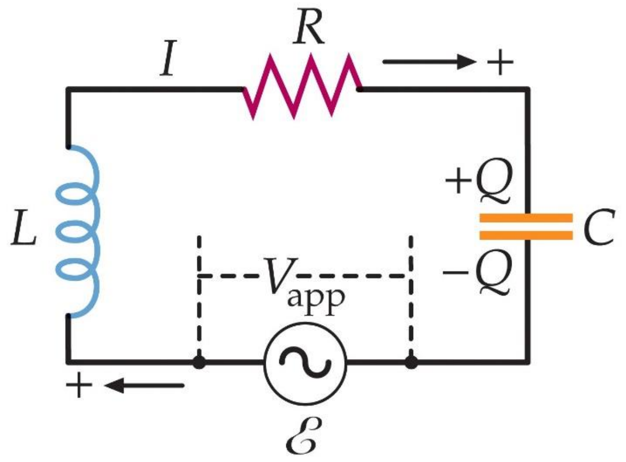
$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T V_0^2 \frac{t^4}{T^4} dt} = \sqrt{\frac{1}{T} V_0^2 \frac{1}{T^4}\frac{T^5}{5}} = \frac{V_0}{\sqrt{5}}$$

Ref:

Phasor for Evaluation The Current in The RLC Circuit

Examples – RLC Circuit

(a) Please find the phase angle ϕ for the RLC series circuit as a function of angular frequency ω. (b) Identify the value of ϕ at the resonance angular frequency ω_0. (c) Calculate the slope of the phase angle ϕ at the resonance point.



current through all components is in phase

Let the voltage source $\varepsilon\cos(\omega t)$

Assume $V_L > V_C$

Current shall be $I = I_0\cos(\omega t - \phi)$

On R, the ac voltage is in phase with the current.

On L ($X_L = \omega L$), the ac voltage leads the current by a phase of $\pi/2$.

On C ($X_C = \frac{1}{\omega C}$), the ac voltage lags behind the current by a phase of $\pi/2$.

Ref:

Phasor for Evaluation The Current in The RLC Circuit

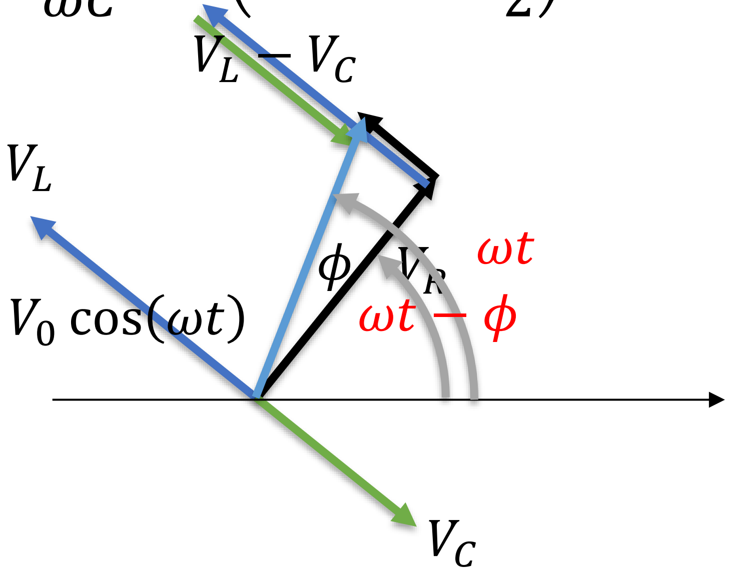
Examples – RLC Circuit

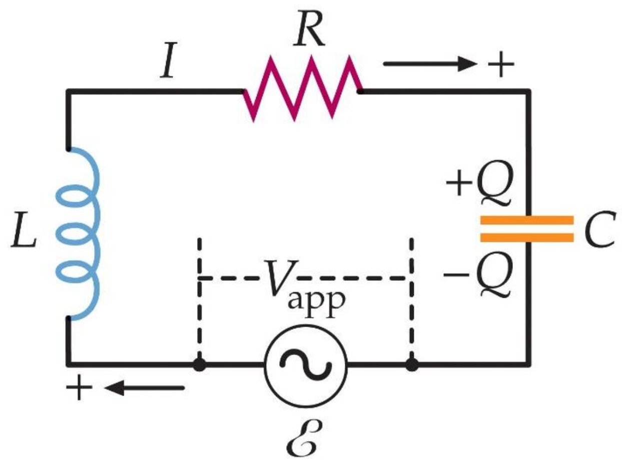
(a) Please find the phase angle ϕ for the RLC series circuit as a function of angular frequency ω. (b) Identify the value of ϕ at the resonance angular frequency ω_0. (c) Calculate the slope of the phase angle ϕ at the resonance point.

$$V_R = RI_0\cos(\omega t - \phi)$$

$$V_L = \omega L I_0 \cos\left(\omega t - \phi + \frac{\pi}{2}\right)$$

$$V_C = \frac{I_0}{\omega C}\cos\left(\omega t - \phi - \frac{\pi}{2}\right)$$





Ref:

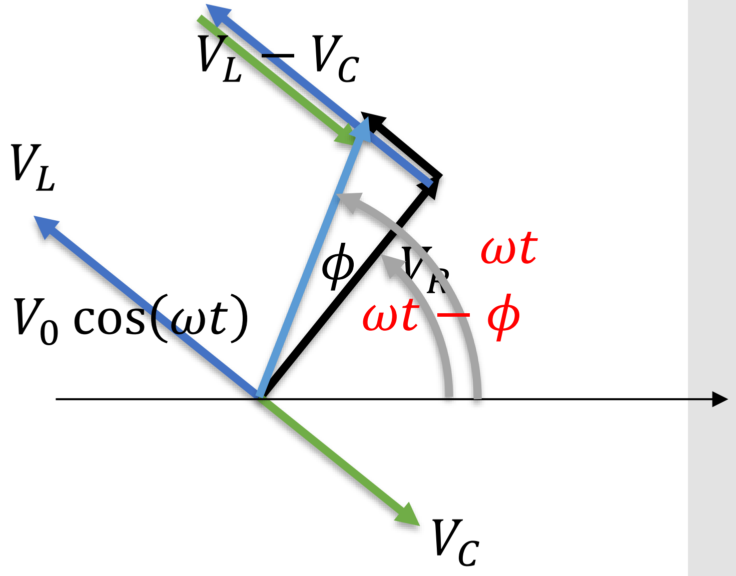
Phasor for Evaluation The Current in The RLC Circuit

Examples – RLC Circuit

(a) Please find the phase angle ϕ for the RLC series circuit as a function of angular frequency ω. (b) Identify the value of ϕ at the resonance angular frequency ω_0. (c) Calculate the slope of the phase angle ϕ at the resonance point.

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{\omega L I_0 - \frac{I_0}{\omega C}}{R I_0} = \frac{X_L - X_C}{R}$$

$$= \frac{\omega^2 CL - 1}{\omega RC}$$



Resonance at $\omega_0^2 = \frac{1}{LC}$

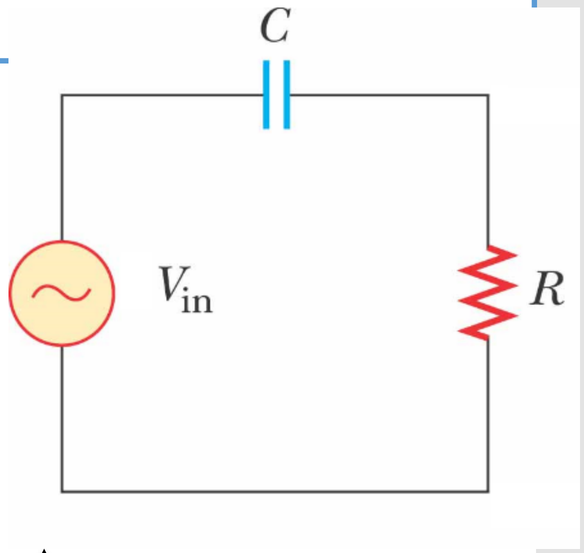
$\tan\phi_0 = 0, \phi_0 = 0$

Ref:

Examples – RC Circuit

Phasor for Evaluation The Current in The RC Circuit

A resistor R and a capacitor C are in series connected with a voltage generator of $V_0\cos(\omega t)$. Find the current in the circuit. Please find the maximum potential drop across the capacitor.



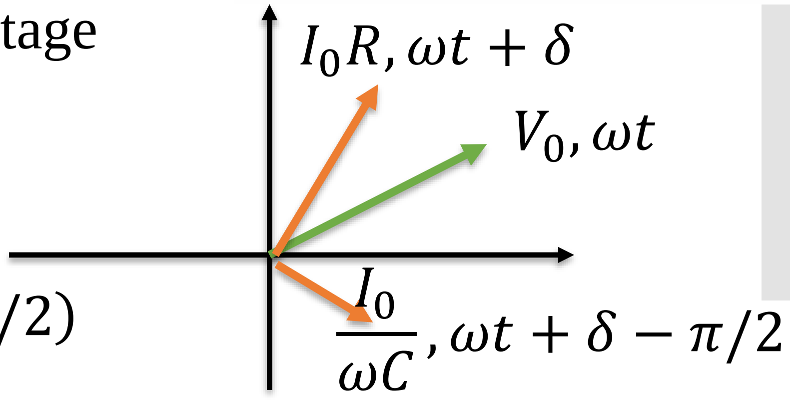
The current leads an advanced phase of $\pi/2$ in comparison with the voltage across the capacitor.

The current will have a phase of $+\delta$: $I = I_0\cos(\omega t + \delta)$.

The current is the same in the series connected circuit so we work on the addition of voltage vectors.

The voltage on R: $I_0 R\cos(\omega t + \delta)$

The voltage on C: $I_0\frac{1}{\omega C}\cos(\omega t + \delta - \pi/2)$

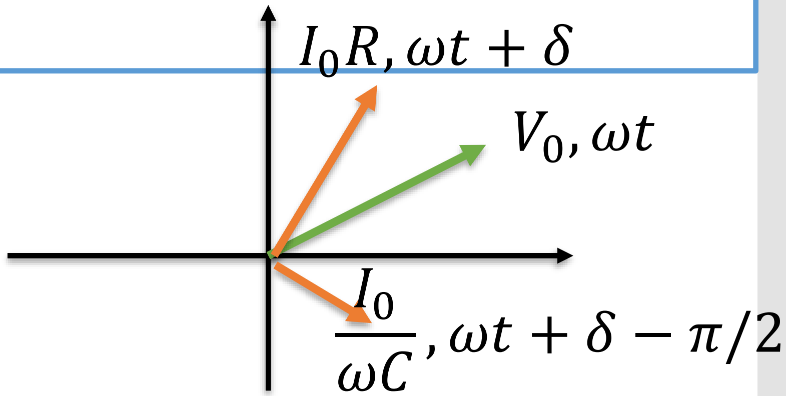


Ref:

Phasor for Evaluation The Current in The RC Circuit

Examples – RC Circuit

A resistor R and a capacitor C are in series connected with a voltage generator of $V_0\cos(\omega t)$. Find the current in the circuit. Please find the potential drop across the capacitor .



$$V_0^2 = I_0^2R^2 + \frac{I_0^2}{\omega^2C^2}$$

$$I_0 = \frac{V_0}{\sqrt{R^2+1/\omega^2C^2}}$$

$$I(t) = \frac{V_0}{\sqrt{R^2+1/\omega^2C^2}}\cos(\omega t+\delta)\,,\tan\delta = \frac{1/\omega C}{R}$$

$$V_C(t) = \frac{1}{\omega C}\frac{V_0}{\sqrt{R^2+1/\omega^2C^2}}\cos(\omega t+\delta-\pi/2)$$

$$V_R(t) = R\frac{V_0}{\sqrt{R^2+1/\omega^2C^2}}\cos(\omega t+\delta)$$

Ref:

Phasor for Evaluation The Current in The RC Circuit

Examples – RC Circuit

A resistor R and a capacitor C are in series connected with a voltage …

$$V_C(t) = \frac{1}{\omega C}\frac{V_0}{\sqrt{R^2 + 1/\omega^2 C^2}}\cos(\omega t + \delta - \pi/2)$$

$$V_R(t) = R\frac{V_0}{\sqrt{R^2 + 1/\omega^2 C^2}}\cos(\omega t + \delta)$$

$$\omega \gg 1 \rightarrow V_C(t) \rightarrow 0, V_R(t) \rightarrow V_0 \cos(\omega t + \delta)$$

The RC circuit is used as an RC filter. When the frequency is much higher, the voltage is mainly dropped on the resistor. The voltage on the resistor is used as a high pass filter.

$$\omega \ll 1 \rightarrow V_C(t) \rightarrow V_0 \cos(\omega t + \delta - \pi/2), V_R(t) \rightarrow 0$$

In contrast, the voltage on the capacitor is used as a low pass filter.

Ref:

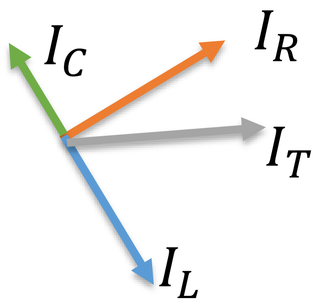
Examples

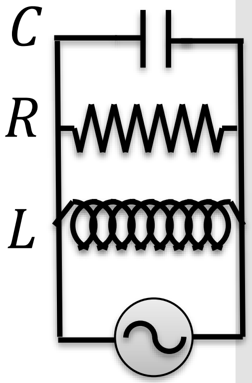
The right figure shows a parallel RLC circuit. The instantaneous voltages across each of the three circuit elements are $V_0 \cos(\omega t)$. (a) Please calculate the current delivered from the source. (b) Please calculate the phase angle between the source voltage and the total current.

$$I_R = \frac{V_0 \cos(\omega t)}{R}$$

$$I_L = \frac{V_0 \cos(\omega t - \pi/2)}{\omega L}$$

$$I_C = \frac{V_0 \cos(\omega t + \pi/2)}{1/\omega C}$$





$$I_T = |I_T| = \sqrt{\left(\frac{V_0}{R}\right)^2 + \left(\frac{V_0}{\omega L} - \omega C V_0\right)^2} = V_0 \sqrt{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{\omega L} - \omega C\right)^2}$$

Examples

The right figure shows a parallel RLC circuit. The instantaneous voltages across each of the three circuit elements are $V_0\cos(\omega t)$. (a) Please calculate the current delivered from the source. (b) Please calculate the phase angle between the source voltage and the total current.

$$I_T(t) = V_0\sqrt{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{\omega L} - \omega C\right)^2}\cos(\omega t - \phi)$$

$$\tan\phi = \frac{\frac{1}{\omega L} - \omega C}{1/R} = \frac{\frac{1}{X_L} - \frac{1}{X_C}}{1/R}$$