Chapter 25 Capacitance & Dielectrics

Physics II – Part I
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Leyden Jar

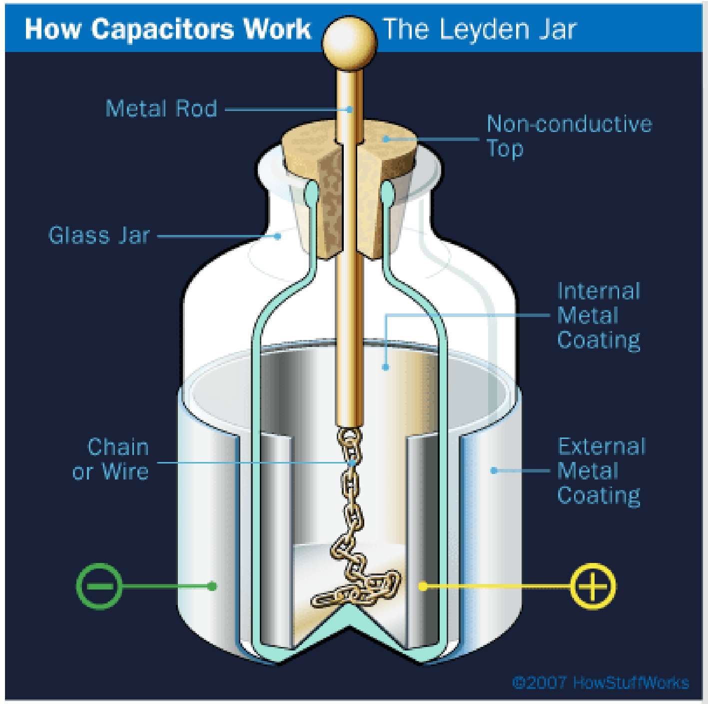
History

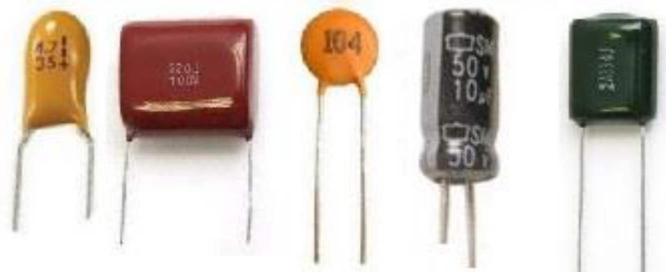
1745 AD – Ewald Georg von Kleist (German)
invent, not published

1745 AD – Pieter van Musschenbroek (Durch)
Dutch professor at the University of Leyden

1800 AD – Michael Faraday (British)
initiate the application of capacitors

1920 AD – practical and commonly used





Types of Capacitors: ceramic capacitors, aluminum electrolyte capacitors, tantalum capacitors, polyester capacitors, polypropylene capacitors, …

http://www.learningaboutelectronics.com/Articles/Types-of-capacitors

Ref: http://electronics.howstuffworks.com/capacitor3.htm, http://www.circuitstoday.com/capacitors-invention-history-and-the-story-of-leyden-jar

Definition & Calculation of Capacitance of a Capacitor

Capacitance

A capacitor is a device consisting of two conductors that can carry equal and opposite charges. The medium between the two conductors is an insulator which is called a dielectric material.

In 1778, Alessandro Volta (Italian) discovered that electrical potential in a capacitor is proportional to the electrical charge in it.

$$V = \frac{Q}{C} \rightarrow C = \frac{Q}{V}$$

The unit of capacitance is known as "jas" before 1872. In 1872, the SI units are changed to "Volt, Ampere, Coulomb, Ohm and Farad".

The unit of capacitance is farad (F). 1 F = 1 C/V; $1\mu F = 10^{-6}F$; $1pF = 10^{-12}F$.

Ref: https://www.famousscientists.org/alessandro-volta/, http://ethw.org/Capacitors

Capacitance of a Parallel Plate Capacitor

Calculation of Capacitance

The two charges Q and –Q are placed on two conductors of a capacitor.

Use Gauss's law to obtain the electric field inside the capacitor.

Integrate to get the voltage difference between the two conductors.

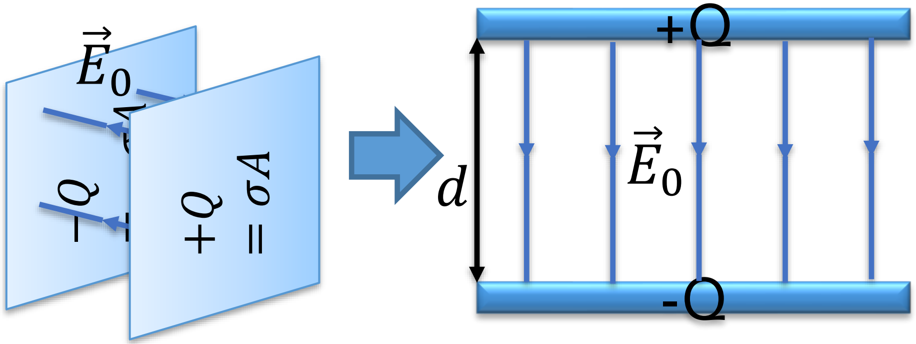
The capacitance is equal to the charge Q divided by the derived voltage.

$$E(2A) = 4\pi k(\sigma A) \rightarrow E = 2\pi k\sigma$$

$$E_{net} = 2 \times 2\pi k\sigma$$

$$V = 4\pi k\sigma d = \frac{4\pi kQd}{A}$$

$$C = \frac{Q}{V} = \frac{A}{4\pi kd} = \frac{A\varepsilon_0}{d}$$



Calculation of Capacitance

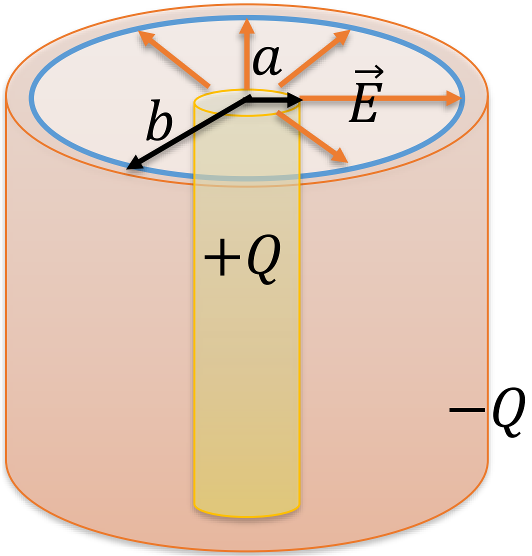
Capacitance of a Cylindrical and a Spherical Capacitors

Cylindrical Capacitor

$$2\pi r L E = 4\pi k Q \rightarrow E = \frac{2kQ}{rL}$$

$$V = -\int_b^a \frac{2kQ}{rL} dr = \frac{2kQ}{L} \ln\left(\frac{b}{a}\right)$$

$$C = \frac{Q}{V} = \frac{L}{2k \ln(b/a)}$$

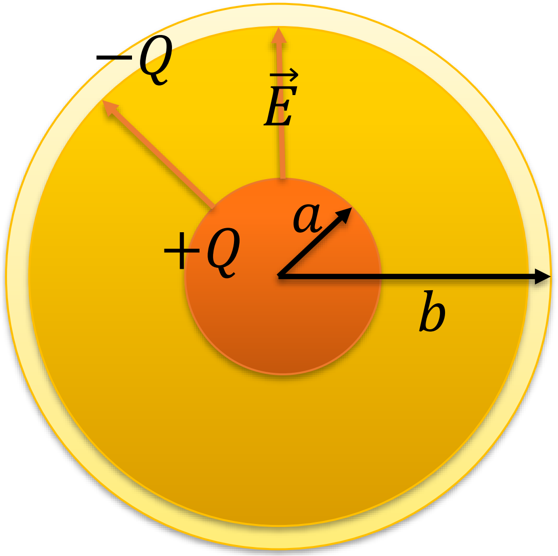


Spherical Capacitor

$$4\pi r^2 E = 4\pi k Q \rightarrow E = \frac{kQ}{r^2}$$

$$V = -\int_b^a \frac{kQ}{r^2} dr = kQ\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$C = \frac{Q}{V} = \frac{ab}{k(b-a)}$$



Self Capacitance of a Spherical Conductor

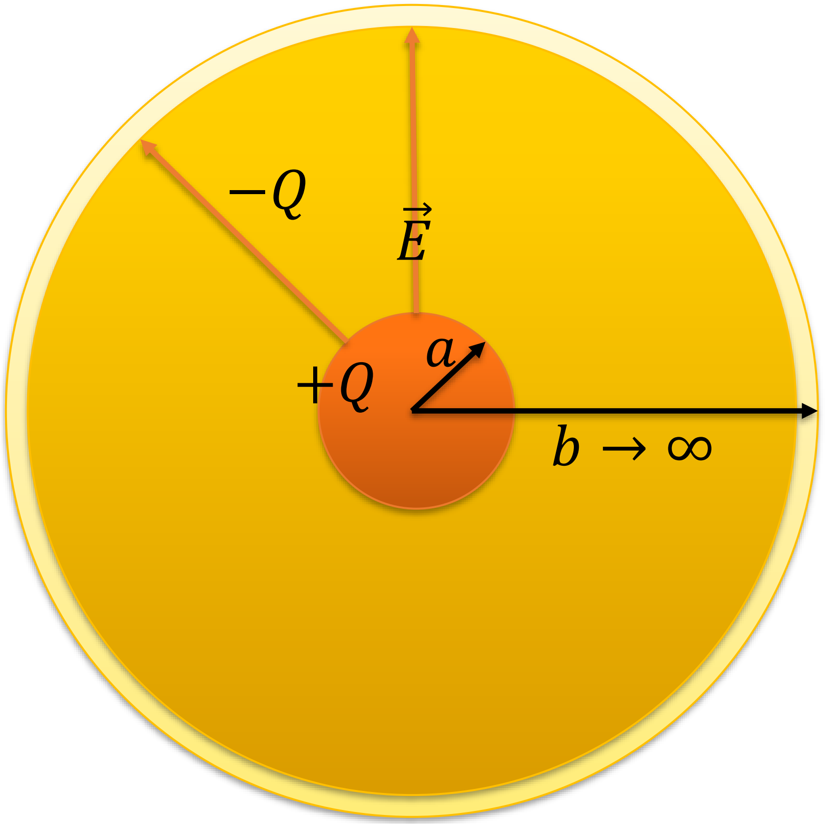
Calculation of Capacitance

The spherical conductor is charged with charge $+Q$ and the imaginary conducting shell with an infinite radius is charged with charge $-Q$.

$$4\pi r^2 E = 4\pi kQ \rightarrow E = \frac{kQ}{r^2}$$

$$V = -\int_{\infty}^{a} \frac{kQ}{r^2} dr = \frac{kQ}{a}$$

$$C = \frac{Q}{V} = \frac{a}{k} = 4\pi\varepsilon_0 a$$



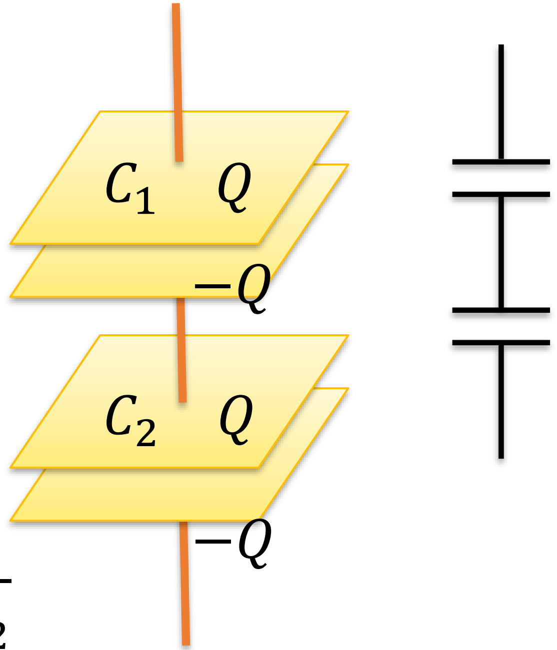
Capacitors Connected in Series

Equivalent Capacitance

For a series connection, the charge induced in the inner connection plate is equal to that placed on the outside plate.

Each capacitor possesses its own voltage and charge with a relation to its capacitance.

The voltage across the two capacitors is summed together.



$$C_1 = \frac{Q}{V_1} \rightarrow V_1 = \frac{Q}{C_1}$$

$$C_2 = \frac{Q}{V_2} \rightarrow V_2 = \frac{Q}{C_2}$$

$$V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2} \rightarrow \frac{1}{Q/V} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

Capacitors Connected in Parallel

Equivalent Capacitance

For a parallel connection, the voltage difference across the two capacitors is the same.

Each capacitor possesses its own voltage and charge with a relation to its capacitance.

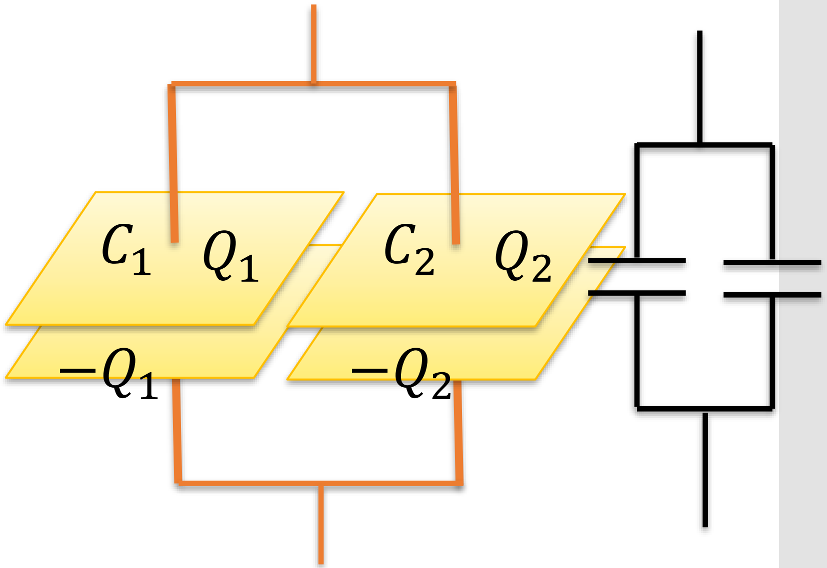
The net charge on one side of the two capacitors is summed together.

$$V_1 = V_2 = V$$

$$C_1 = \frac{Q_1}{V} \rightarrow Q_1 = C_1 V \quad Q_2 = C_2 V$$

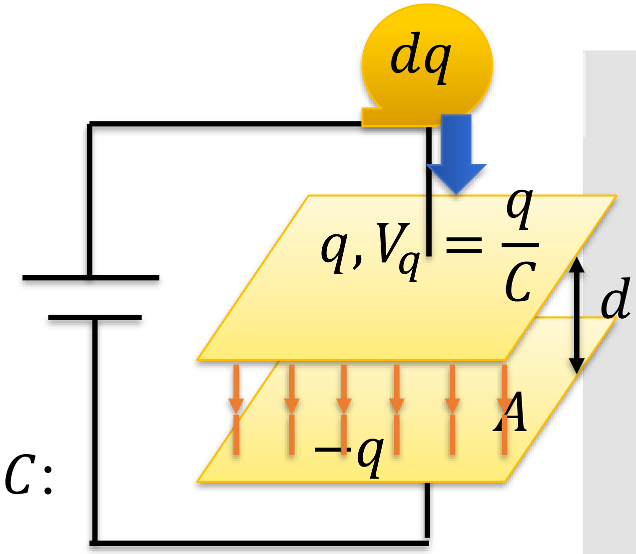
$$Q = Q_1 + Q_2 = C_1 V + C_2 V$$

$$\frac{Q}{V} = C_1 + C_2 \rightarrow C = C_1 + C_2$$



Stored Energy

Energy Stored in a Charged Capacitor & in Electric Field



When the parallel capacitor is charged up to charge q, the voltage across the capacitor is

$$V(q) = \frac{q}{C} \rightarrow dU = V(q)dq = \frac{q}{C}dq$$

Total energy when charged up to Q and $V = Q/C$:

$$U = \int_0^Q \frac{q}{C}dq = \frac{Q^2}{2C} = \frac{1}{2}QV = \frac{1}{2}CV^2$$

The energy of electric field inside the parallel capacitor is the same as the capacitor charging energy. Assume an energy density of electric field as $u_E = U/V$. Change energy to electric field $(Q/A\varepsilon_0)$ expression.

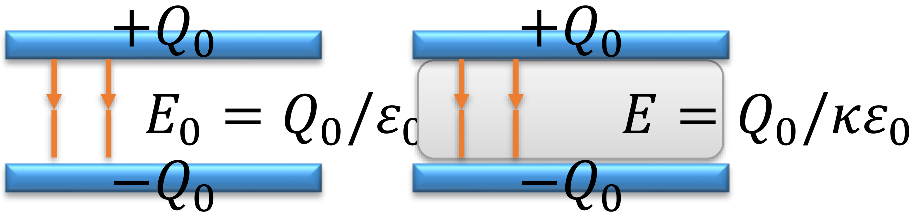
$$U = \frac{Q^2}{2C} \rightarrow u_E Ad = \frac{A^2\varepsilon_0^2}{2C}\left(\frac{Q}{A\varepsilon_0}\right)^2 \rightarrow u_E Ad = \frac{A^2\varepsilon_0^2}{2A\varepsilon_0/d}\left(\frac{Q}{A\varepsilon_0}\right)^2$$

$$\rightarrow u_E = \frac{\varepsilon_0}{2}\left(\frac{Q}{A\varepsilon_0}\right)^2 = \frac{\varepsilon_0 E^2}{2}$$

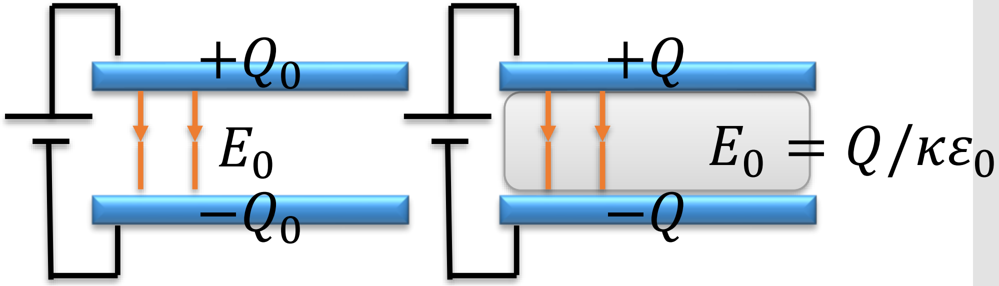
The Role of Dielectrics

Dielectrics: change the vacuum permittivity ε_0 to $\kappa\varepsilon_0 = \varepsilon$ ($\kappa > 1$), increase the charge storage capability

Constant charge condition, the electric field inside is reduced to $Q_0/\kappa\varepsilon_0$



Constant voltage condition, the charge is increased to κQ_0



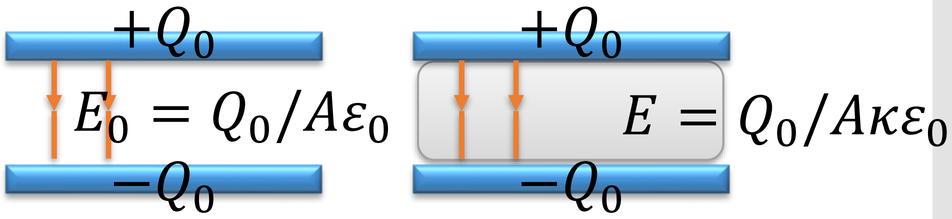
$$\frac{V_0}{d} = E_0 = \frac{Q_0}{\varepsilon_0} = \frac{Q}{\kappa\varepsilon_0} \rightarrow Q = \kappa Q_0$$

Material	Air	Glass	Mica	Al2O3	Polystyrene	HfO2 or ZrO2
Dielectric Constant κ	1.00059	5.6	5.4	9.1	2.55	25

Bound Charge

The Role of Dielectrics

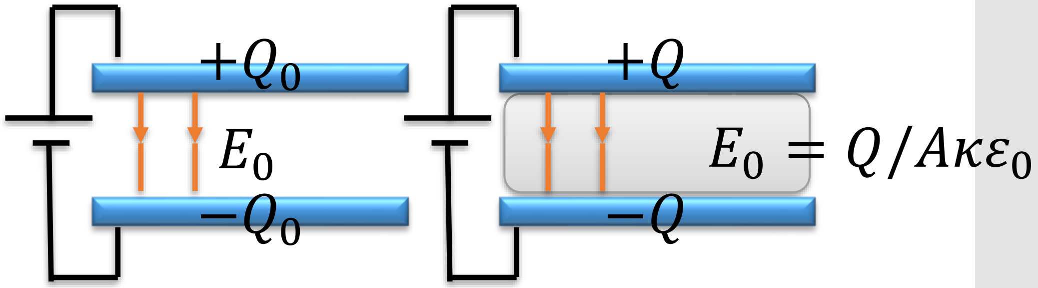
Constant charge condition, the bond charge is used to decrease inner electric field



$$E_0 \to E \Rightarrow Q_0 \to \frac{Q_0}{\kappa} \Rightarrow Q_0 \to Q_0 - Q_b$$

$$Q_0 - Q_b = \frac{Q_0}{\kappa} \qquad Q_b = Q_0\frac{\kappa - 1}{\kappa}$$

Constant voltage condition, additional charge is supplied to balance the bound charge



$$E_0 = \frac{Q_0}{A\varepsilon_0} = \frac{Q}{A\kappa\varepsilon_0} \to Q_0 = \frac{Q}{\kappa}$$

$$Q = Q_0 + Q_b$$

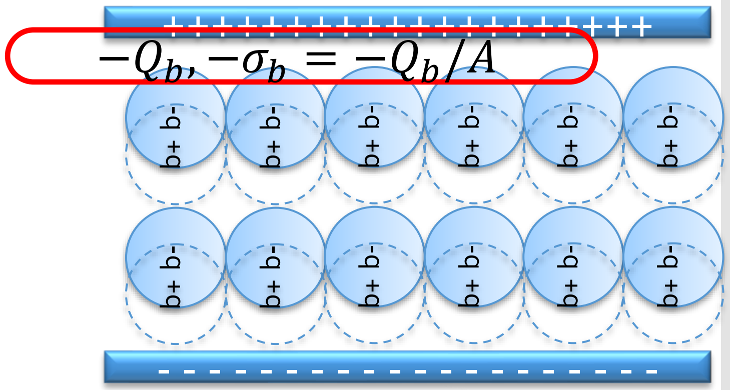
$$\kappa Q_0 = Q_0 + Q_b = Q_0 \qquad Q_b = (\kappa - 1)Q_0$$

Energy Stored in The Presence of Dielectrics

The Role of Dielectrics

Energy stored in the capacitor and the energy density of an electric field change with dielectrics replacing the vacuum. Just use $\varepsilon = \kappa\varepsilon_0$ to replace ε_0. At a constant voltage, the stored energy is estimated to be $U = \frac{1}{2}CV_0^2$ for a parallel plate capacitor, $U = \frac{1}{2}\frac{A\varepsilon}{d}V_0^2$.

Electric field with constant strength, E_0:



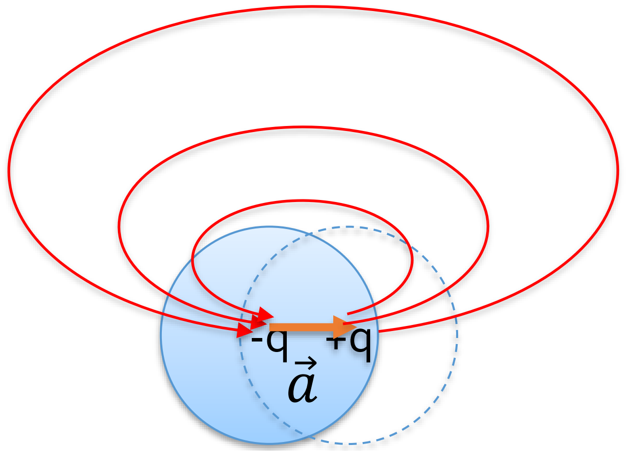
$$u_E = \frac{1}{2}\varepsilon_0 E_0^2 \rightarrow u_E = \frac{1}{2}\varepsilon E_0^2$$

Additional energy $\frac{1}{2}(\kappa - 1)\varepsilon_0 E_0^2$ stored in separating electrons and holes in atoms of dielectrics.

Electric Dipole

Electric Potential and Torque of Dipoles in Electric Field

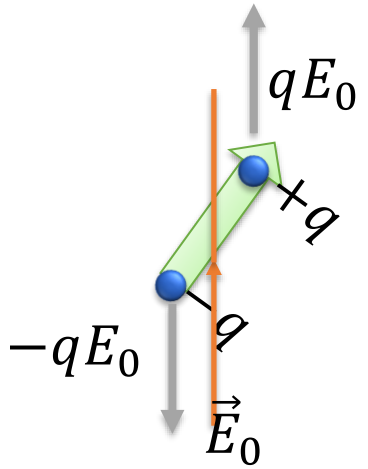
Distortion of electron cloud of an atom. The vector of electric dipole moment $\vec{p}, p = qa$



Electric dipole moment in electric field

$$\vec{\tau} = \frac{1}{2}\vec{a} \times (q\vec{E}_0) + \left(-\frac{1}{2}\vec{a}\right) \times (-q\vec{E}_0)$$

$$\vec{\tau} = (q\vec{a}) \times \vec{E}_0 = \vec{p} \times \vec{E}_0 = pE_0 \sin(\theta)$$



To store potential energy, a negative torque must be exert, $\tau = -pE_0 \sin(\theta)$

$$dU = -\tau d\theta = -(-pE_0 \sin(\theta))d\theta = pE_0 \sin(\theta)\, d\theta$$

$$U = \int_{\pi/2}^{\theta} pE_0 \sin(\theta)\, d\theta = -pE_0 \cos(\theta) = -\vec{p} \cdot \vec{E}_0$$

Charge Conservation, Redistribution

Examples

Two charged capacitors are carefully connected in parallel. Please calculate the voltage across the capacitors and the charge on the two capacitors.

Consider the charge conservation

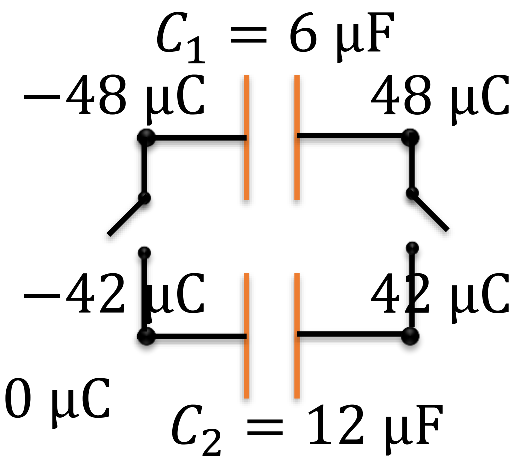
$$Q_{net} = 90\ \mu\text{C}$$

The same voltage difference across the two capacitors

$$V_1 = V_2 = V \rightarrow C_1 = \frac{Q_1}{V}, C_2 = \frac{Q_2}{V}, Q_1 + Q_2 = 90\ \mu\text{C}$$

$$6V + 12V = 90 \rightarrow V = 5\ \text{V}$$

$$Q_1 = 30\ \mu C, Q_2 = 60\ \mu C$$



Examples

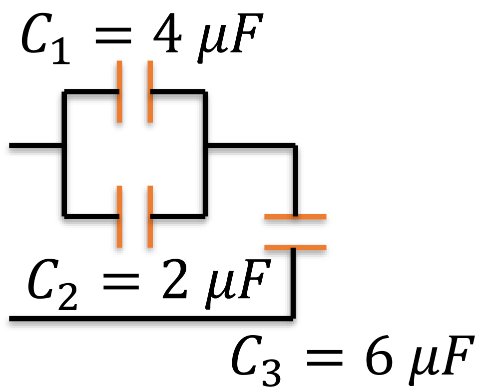
Please calculate the equivalent capacitance of the capacitor circuit.

Use the parallel/series connection rules for calculation

$$C_{12,equi} = C_1 + C_2 = 6\ \mu F$$

$$\frac{1}{C_{net}} = \frac{1}{C_{12,equi}} + \frac{1}{C_3} = \frac{1}{6} + \frac{1}{6}$$

$$C_{net} = 3\ \mu F$$



Equivalent Capacitance

Examples

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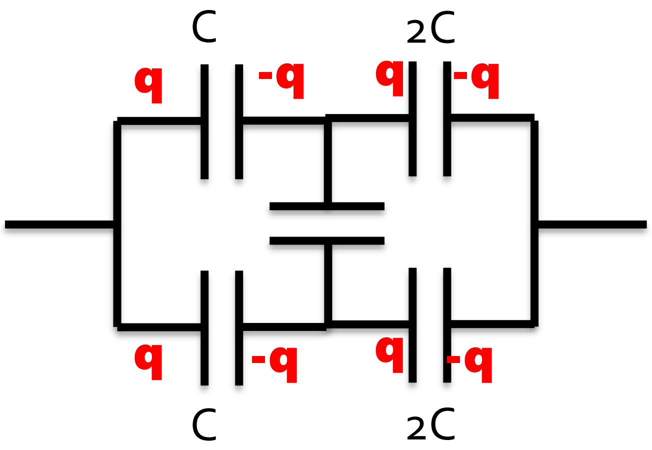

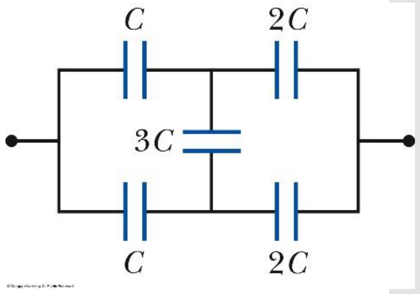

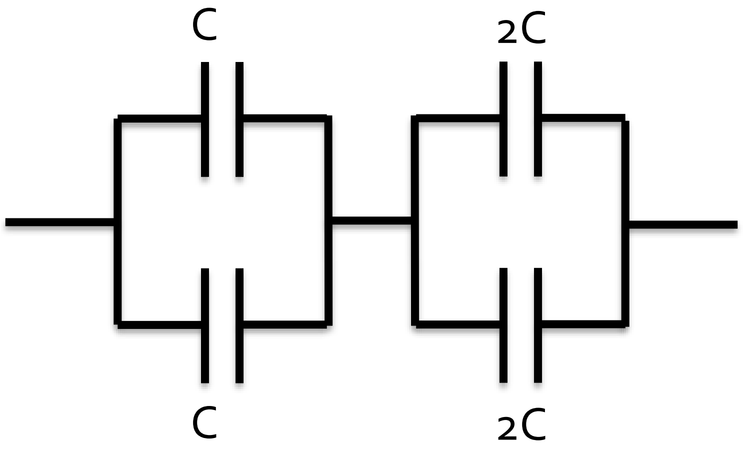

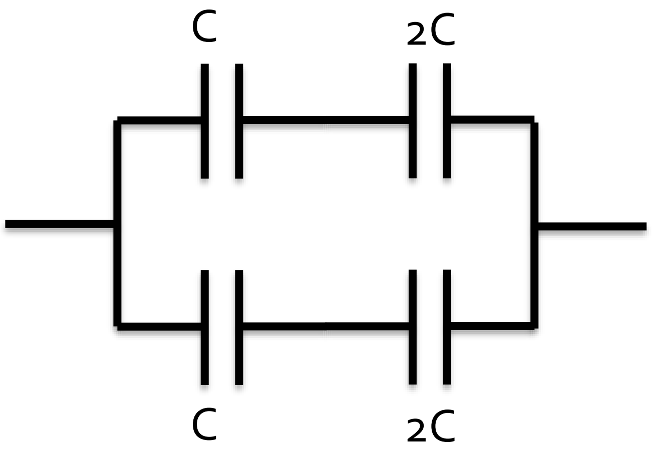

Equivalent Capacitance

Examples

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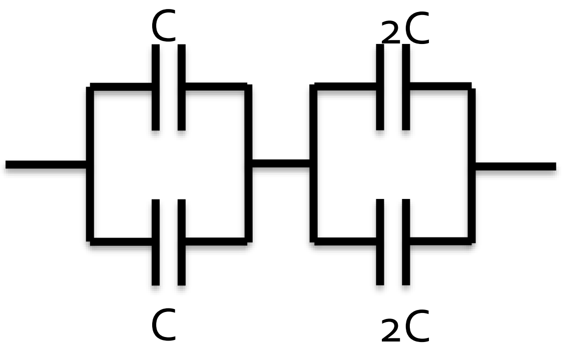
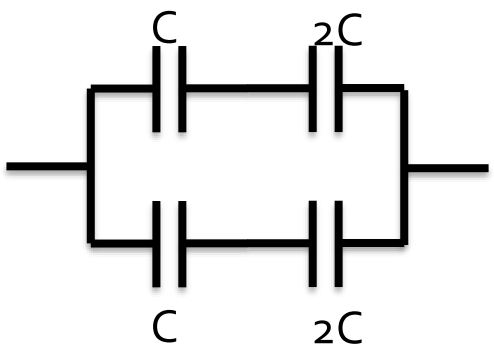
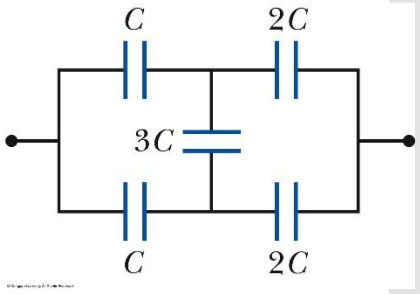






$$\frac{1}{C_t} = \frac{1}{C + C} + \frac{1}{2C + 2C} = \frac{3}{4C}$$

$$C_t = \frac{4}{3}C$$

$$\frac{1}{C} = \frac{1}{C} + \frac{1}{2C} = \frac{3}{2C}$$

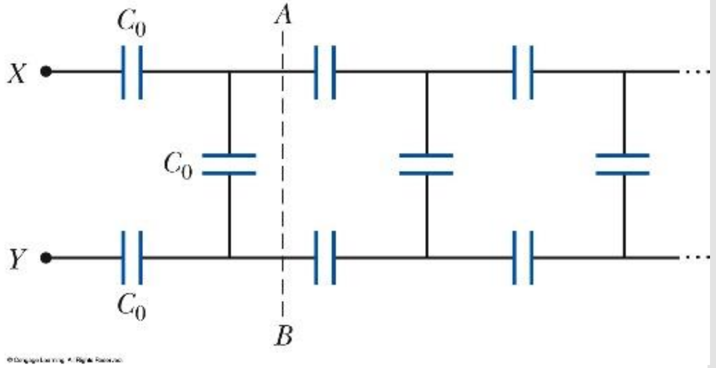
$$C = \frac{2}{3}C$$

$$C_t = \frac{2}{3}C + \frac{2}{3}C = \frac{4}{3}C$$

Equivalent Capacitance

Examples

Some physical systems such as microwave waveguide and the axon of a nerve cell possessing capacitance continuously distributed over space can be modeled as an infinite array of discrete circuit elements. To analyze an infinite array, determine the equivalent capacitance C between terminals X and Y of the infinite set of capacitors shown in the figure. Each capacitor has capacitance C_0.



$$\frac{1}{C_{total}} = \frac{1}{C_0} + \frac{1}{C_0 + C_{total}} + \frac{1}{C_0}$$

$$\frac{1}{C_{total}} = \frac{C_0 + 2(C_0 + C_{total})}{C_0(C_0 + C_{total})}$$

$$C_0(C_0 + C_{total}) = (3C_0 + 2C_{total})C_{total}$$

$$2C_{total}^2 + 2C_0C_{total} - C_0^2 = 0 \qquad C_{total} = \frac{-2 \pm \sqrt{12}}{4} C_0$$

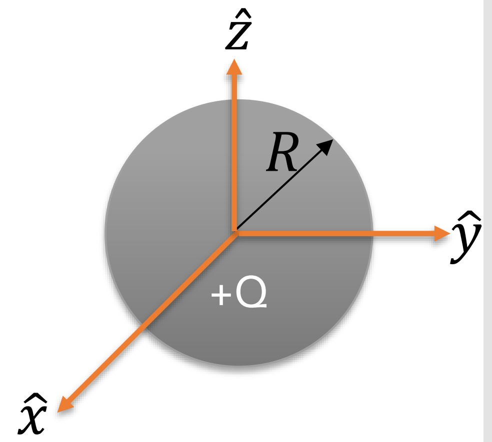
$$C_{total} = \frac{\sqrt{3} - 1}{2} C_0$$

Energy Density of Electric Field

Examples

Please calculate the build up energy for a spherical conductor of radius R charged with net charge of Q.

Put the center of the sphere on the origin of the coordinate system.



Use Gauss's law to obtain the electric field.

$$r > R, E = \frac{kQ}{r^2}$$

Use the energy density of electric field.

$$\xi = \frac{\varepsilon_0}{2} E^2 = \frac{\varepsilon_0 k^2 Q^2}{2r^4}$$

$$U = \int_{r=R}^{\infty} \xi 4\pi r^2 dr = \int_{R}^{\infty} \frac{4\pi\varepsilon_0 k^2 Q^2}{2r^2} dr = \frac{kQ^2}{2R} = \frac{1}{2} Q \frac{kQ}{R} = \frac{1}{2} QV$$

Parallel or Series Connection for Capacitance Calculation

Examples

A parallel-plate capacitor has square plates of area A and a separation of d. A dielectric slab of dielectric constant κ has the same area A and a thickness of d. (a) What is the capacitance without the dielectric? (b) What is the capacitance with the dielectric? (c) What is the capacitance if a dielectric slab having a thickness of $3d/4$ is inserted into the capacitor and attached to one metal plate of the capacitor?

(a)

Use Gauss's law to obtain $E = 4\pi kQ/A$ and obtain the voltage

$$V = \frac{4\pi kQd}{A} \rightarrow C = \frac{Q}{V} = \frac{A}{4\pi kd} = \frac{A\varepsilon_0}{d}$$

(b)

Change ε_0 to $\kappa\varepsilon_0$

$$C = \frac{A\kappa\varepsilon_0}{d}$$

Parallel or Series Connection for Capacitance Calculation

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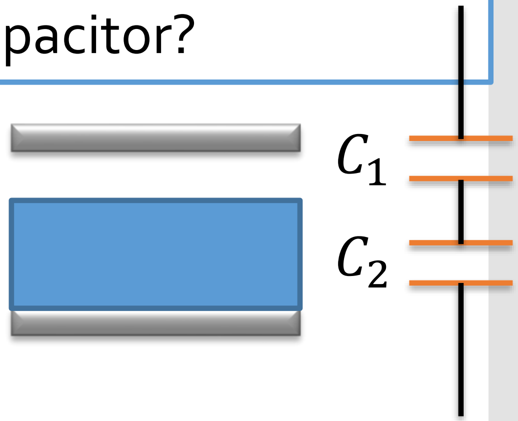
(c)

$$C_1 = \frac{A\varepsilon_0}{d/4} \quad C_2 = \frac{A\kappa\varepsilon_0}{3d/4}$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{d/4}{A\varepsilon_0} + \frac{3d/4\kappa}{A\varepsilon_0} = \frac{(\kappa+3)d/4\kappa}{A\varepsilon_0}$$

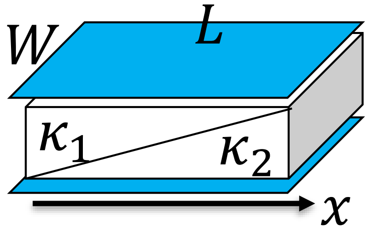
$$C = \frac{4\kappa A\varepsilon_0}{(\kappa+3)d}$$



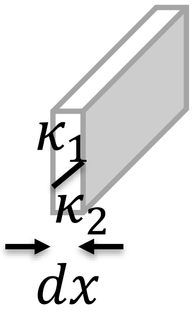
Parallel or Series Connection for Capacitance Calculation

Examples

A parallel plate capacitor with plates of area $L \times W$ and separation t has the region between its plates filled with wedges of two dielectric materials. Assume t is much less than both W and L. (a) Please determine its capacitance.



The thickness of the κ_1 dielectrics decreases as $t\frac{(L-x)}{L}$. The thickness of the κ_2 dielectrics increases as $t\frac{x}{L}$.



For a short stripe of a width dx, the series connected capacitance C is $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$, where $C_1 = \frac{A\varepsilon}{d_1} = \frac{Wdx\kappa_1\varepsilon_0}{t(L-x)/L}$ and $C_2 = \frac{A\varepsilon}{d_2} = \frac{Wdx\kappa_2\varepsilon_0}{tx/L}$.

$$C = \frac{W\varepsilon_0 dx}{t\left(\frac{1}{\kappa_1} + \frac{x}{L}\left(\frac{1}{\kappa_2} - \frac{1}{\kappa_1}\right)\right)}$$

Parallel or Series Connection for Capacitance Calculation

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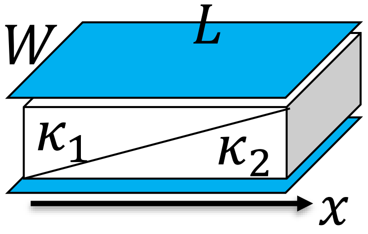
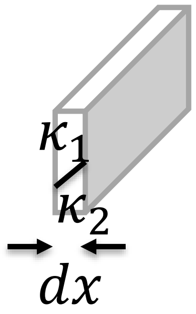
A parallel plate capacitor with plates of area $L \times W$ and separation t has the region between its plates filled with wedges of two dielectric materials. Assume t is much less than both W and L. (a) Please determine its capacitance.

$$C = \frac{W\varepsilon_0 dx}{t\left(\frac{1}{\kappa_1} + \frac{x}{L}\left(\frac{1}{\kappa_2} - \frac{1}{\kappa_1}\right)\right)}$$



All the stripes are parallel connected, thus

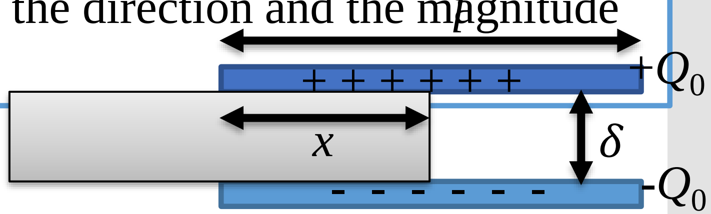
$$C_{total} = \int_0^L \frac{W\varepsilon_0 dx}{t\left(\frac{1}{\kappa_1} + \frac{x}{L}\left(\frac{1}{\kappa_2} - \frac{1}{\kappa_1}\right)\right)} = \frac{W\varepsilon_0 L}{t\left(\frac{1}{\kappa_2} - \frac{1}{\kappa_1}\right)} \ln\left(\frac{\kappa_1}{\kappa_2}\right)$$



Examples

Calculation of Force from Energy Stored in a Capacitor

Two square plates of sides l are placed parallel to each other with separation δ, where $\delta \ll l$. The plates carry uniformly distributed static charges $+Q_0$ and $-Q_0$. A block of metal with width l, length l, and thickness slightly less than δ is inserted a distance x into the space between the plates. The charge on the plates remains uniformly distributed. In a static situation, a metal prevents an electric field from penetrating inside it and can be thought of as a perfect dielectric with $\kappa \to \infty$. (a) Calculate the stored energy in the system as a function of x. (b) Find the direction and the magnitude of the force acting on the metallic block.



Inside the metal $E = 0$

In the space without dielectric materials

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q_0}{\varepsilon_0 l^2}$$

The energy density is $$\frac{\varepsilon_0 E^2}{2} = \frac{\varepsilon_0}{2}\left(\frac{Q_0}{\varepsilon_0 l^2}\right)^2 = \frac{Q_0^2}{2\varepsilon_0 l^4}$$

the total energy $$(l(l-x)\delta)\frac{Q_0^2}{2\varepsilon_0 l^4}$$

$$U(x) = \frac{\delta Q_0^2 (l-x)}{2\varepsilon_0 l^3}$$

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Examples

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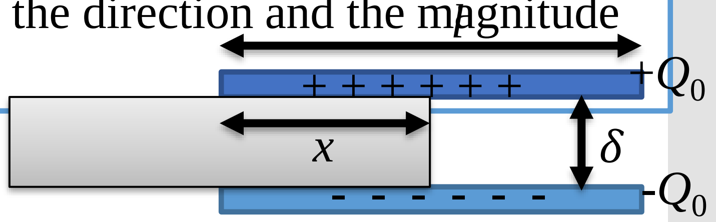


The force is derived from the potential energy.

$$F(x) = -\frac{dU}{dx} = \frac{\delta Q_0^2}{2\varepsilon_0 l^3}$$

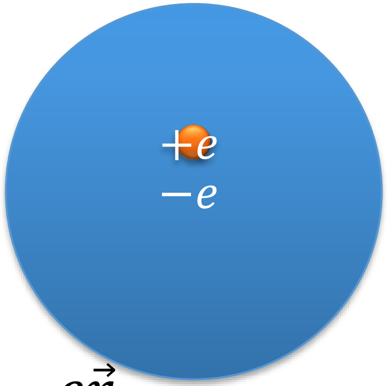
An Atomic Description of Dielectrics

Examples

A hydrogen atom consists of a proton nucleus of charge $+e$ and an electron of charge – e. The charge distribution of the atom is spherically symmetric, so the atom is nonpolar. Consider a model in which the hydrogen atom consists of a positive charge $+e$ at the center of a uniformly charged spherical cloud of radius R and total charge – e. Show that when such an atom is placed in a uniform external field E, the induced dipole moment is proportional to E; that is, $p = \alpha E$, where α is called the polarizability. Please find α.

Take the $-e$ charge as uniformly charged sphere of radius R.

$$-e = \rho \frac{4\pi R^3}{3} \rightarrow \rho = -\frac{3e}{4\pi R^3}$$



For a position with a distance r away from the center, the electric field due to the uniformly charged sphere is $4\pi r^2 E = \rho \frac{4\pi r^3}{3\varepsilon_0} \rightarrow \vec{E} = \frac{\rho \vec{r}}{3\varepsilon_0} = -\frac{e\vec{r}}{4\pi\varepsilon_0 R^3}$

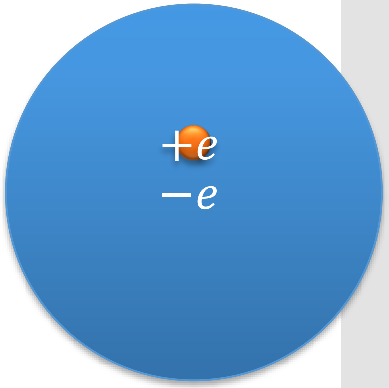
An Atomic Description of Dielectrics

Examples

A hydrogen atom consists of a proton nucleus of charge $+e$ and an electron of charge $-e$. The charge distribution of the atom is spherically symmetric, so the atom is nonpolar. Consider a model in which the hydrogen atom consists of a positive charge $+e$ at the ...

$$\vec{E} = -\frac{e\vec{r}}{4\pi\varepsilon_0 R^3}$$

The electric dipole p is defined as $p = qd = er$.



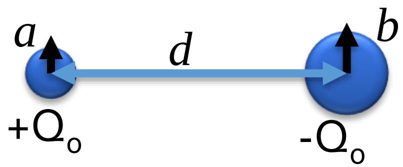
According to the proposed model of $p = \alpha E$, we put $p = er$ and $E = |\vec{E}| = \frac{er}{4\pi\varepsilon_0 R^3}$ to find α.

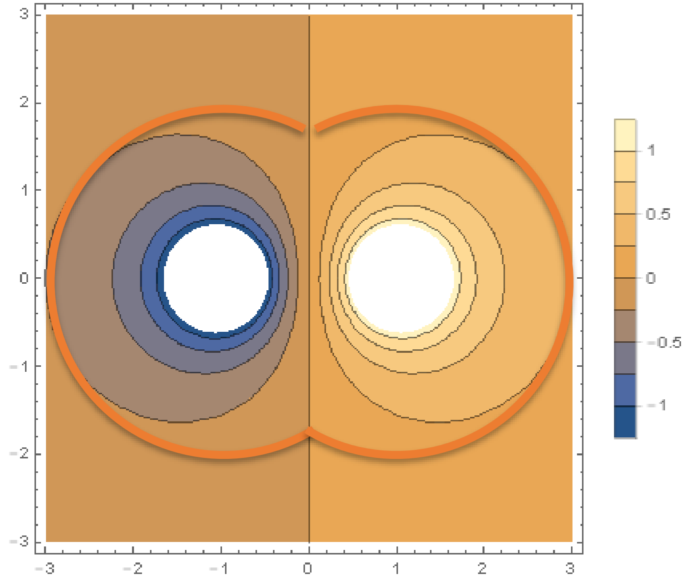
$$er = \alpha\frac{er}{4\pi\varepsilon_0 R^3} \rightarrow \alpha = 4\pi\varepsilon_0 R^3$$

An Atomic Description of Dielectrics

Examples

Two spheres have radii a and b, and their centers are a distance d apart. Show that the capacitance of this system is $C = \frac{4\pi\varepsilon_0}{\frac{1}{a}+\frac{1}{b}-\frac{2}{d}}$, provided d is large compared with a and b. Show that as d approaches infinity, the capacitance reduces to that of two spherical capacitors in series.





The first step is to find a reference position to calculate the potential of the two spheres.

An Atomic Description of Dielectrics

Examples

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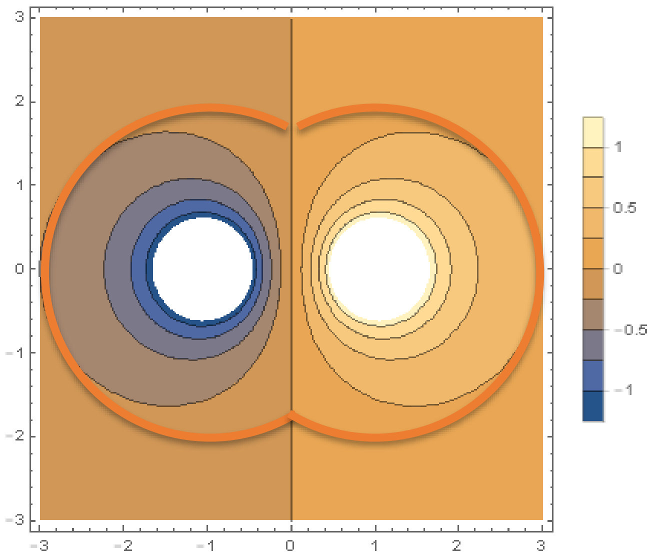
$$V_1 = -\int_{d-b}^{a} \frac{kQ_0}{r^2}\,dr = \frac{kQ_0}{a} - \frac{kQ_0}{d-b}$$

$$V_2 = \int_{d-a}^{b} \frac{kQ_0}{r^2}\,dr = -\frac{kQ_0}{b} + \frac{kQ_0}{d-a}$$

$$\Delta V = V_1 - V_2 = \frac{kQ_0}{a} - \frac{kQ_0}{d-b} + \frac{kQ_0}{b} - \frac{kQ_0}{d-a}$$



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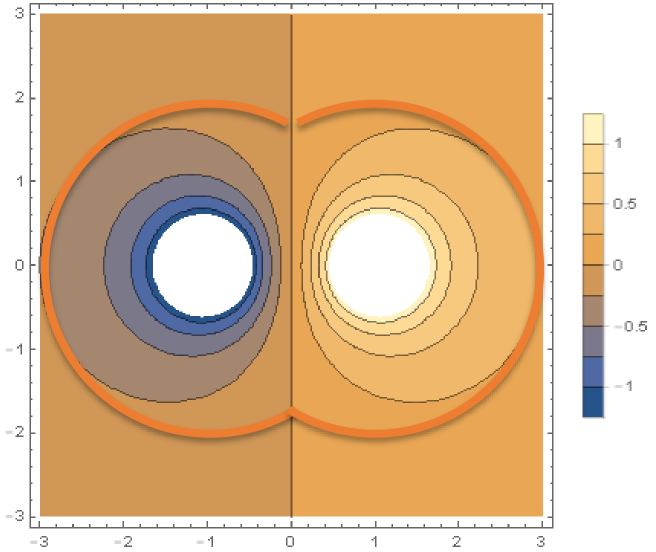
$$\frac{1}{C} = \frac{\Delta V}{Q_0} = \frac{k}{a} + \frac{k}{b} - \frac{k}{d-b} - \frac{k}{d-a}$$

$$C = \frac{1}{k}\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{d-b}-\frac{1}{d-a}} \cong \frac{1}{k}\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{2}{d}}$$

$$d \to \infty, C = \frac{4\pi\varepsilon_0}{\frac{1}{a}+\frac{1}{b}} \to \frac{1}{C} = \frac{1}{4\pi\varepsilon_0 a} + \frac{1}{4\pi\varepsilon_0 b}$$



Self Capacitances Connected in Serial