Chapter 23 Gauss's Law

Physics II – Part I
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Mathematical Work for Natural World

History

1773 AD – Joseph-Louis Lagrange (Italian)
first formulate the law

1813 AD – Carl Friedrich Gauss (German)
follow Lagrange's work

integral form, closed surface:

$$\oiint \vec{E} \cdot d\vec{S} = 4\pi k Q_{enclosed} = \frac{Q_{enclosed}}{\varepsilon_0}$$

differential form (divergence theorem):

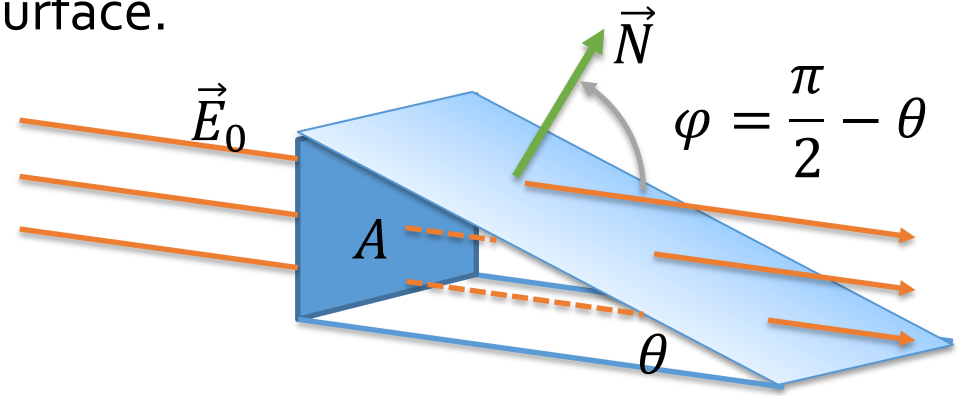
$$\vec{\nabla} \cdot \vec{E} = 4\pi k \rho = \frac{\rho}{\varepsilon_0}$$

Ref: https://en.wikipedia.org/wiki/Gauss%27_law

Electric Flux Through An Open Surface

Electric Flux

The concept of electric flux is like the calculation of the number of electric field lines through the surface.



The flux through the perpendicular surface

$$\Phi_E = AE_0$$

The flux through the inclined surface is the same.

$$\Phi_E = A'E_0 = \frac{A}{\sin(\theta)} E_0 \xrightarrow{\times \cos\left(\frac{\pi}{2} - \theta\right)} AE_0$$

$$\Phi_E = \vec{A} \cdot \vec{E} = A'E_0 \cos(\varphi)$$

Electric Flux Through a Closed Surface

Net Electric Flux

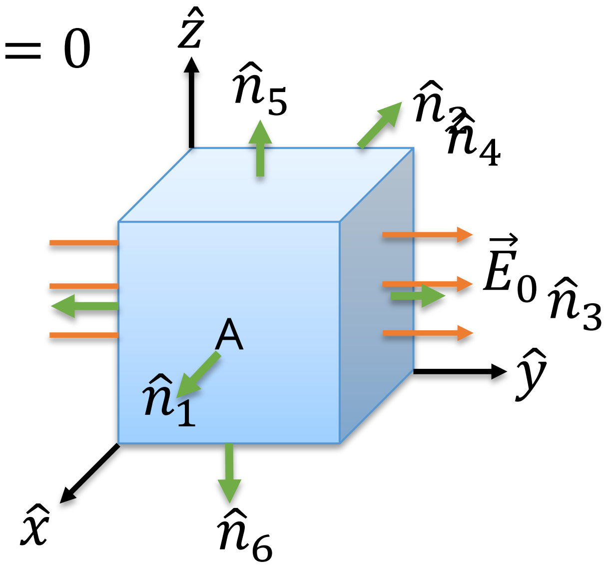
Uniform electric field $\vec{E}_0 = E_0\hat{j}$ through a closed surface of a box.

$$\Phi_1 = \Phi_2 = \Phi_5 = \Phi_6 = \vec{E}_0 \cdot \left(A\hat{n}_{i,i=1,2,5,6}\right) = 0$$

$$\Phi_3 = \vec{E}_0 \cdot \hat{n}_3 = (E_0\hat{j}) \cdot (A\hat{j}) = E_0 A$$

$$\Phi_4 = \vec{E}_0 \cdot \hat{n}_4 = (E_0\hat{j}) \cdot (-A\hat{j}) = -E_0 A$$

$$\Phi_{net} = \sum_{i=1}^{6} \Phi_i = 0$$



Ref:

Electric Flux of a Closed Spherical Surface With Enclosed Charge

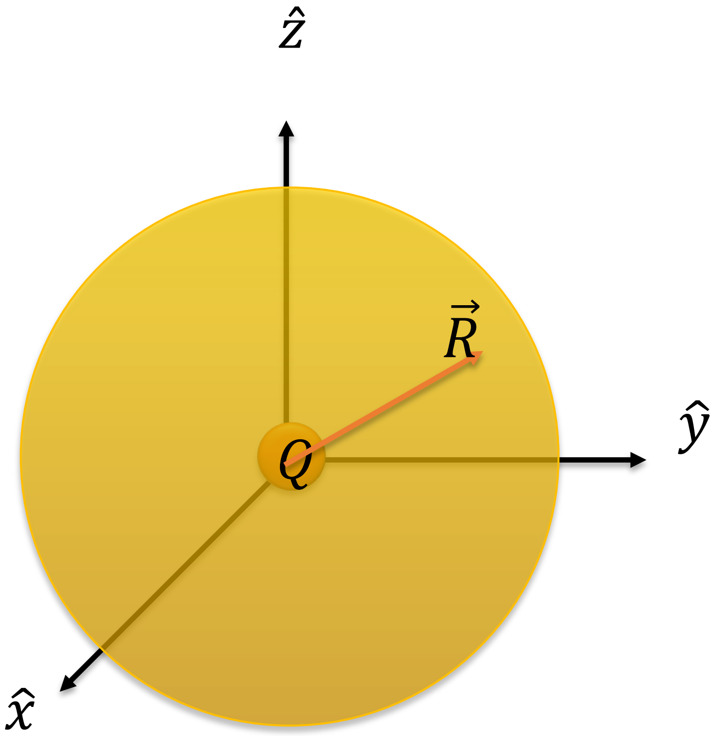
Net Electric Flux of a Point Charge

Electric field of a charge Q on the origin:

$$\vec{E}(\vec{r}) = \frac{kQ}{R^2}\hat{r}, \hat{r} = \hat{R}$$

The infinitesimal surface is

$$d\vec{S} = (Rd\theta\,)(R\sin(\theta)\,d\phi)\hat{r}$$



$$\Phi_{E,net} = \oiint \left(\frac{kQ}{R^2}\hat{r}\right) \cdot \left((Rd\theta\,)(R\sin(\theta)\,d\phi)\hat{r}\right)$$

$$\Phi_{net} = \int_0^{\pi}\int_0^{2\pi} kQ\sin(\theta)\,d\phi d\theta = 4\pi kQ$$

Ref:

Arbitrary Shape of Surface

Gauss's Law

Simple derivation of Gauss's law:

$$\Delta\Phi_E = \vec{E}\cdot\Delta\vec{A}$$

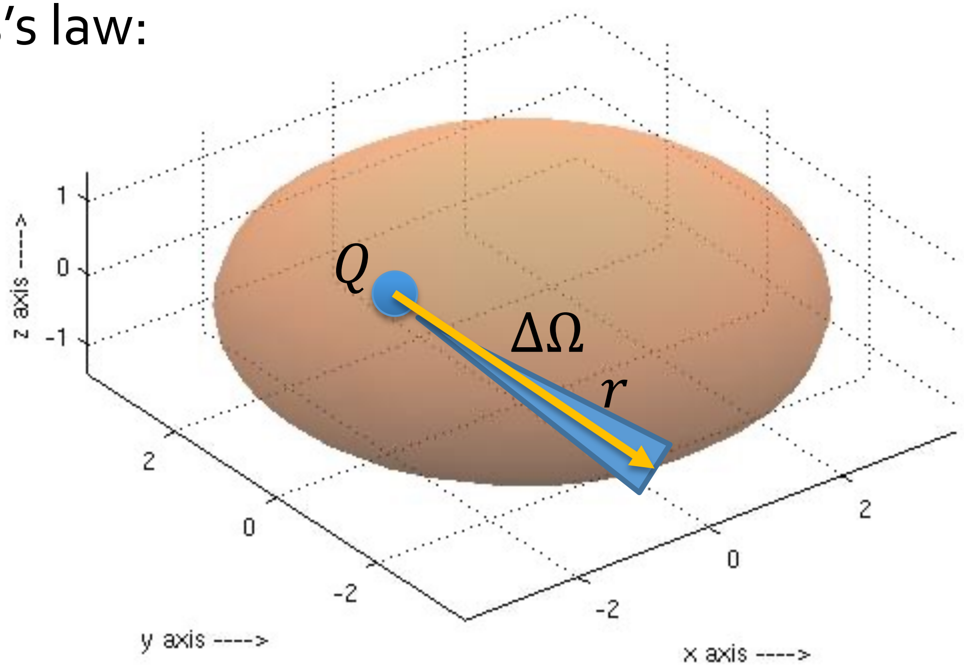
$$\vec{E} = \frac{kQ}{r^2}\hat{r}$$

$$\Delta\vec{A} = \Delta A_{\perp}\hat{r}$$

$$\Delta\Phi_E = \left(\frac{kQ}{r^2}\hat{r}\right)\cdot(\Delta A_{\perp}\hat{r})$$

$$\Delta\Phi_E = kQ\frac{\Delta A}{r^2} = kQ\Delta\Omega$$

$$\Phi_{E,net} = kQ\oint d\Omega = 4\pi kQ$$



Ref:

Calculate Electric Field Using Gauss's Law

Gauss's law can be easily applied in symmetrical objects. Those include plane, cylindrical, and spherical symmetry objects.

Find an imaginary Gauss's plane. Electric fields are equal in magnitude and all perpendicular to the plane.

Sometimes, we can use some zero-flux planes on which the electric fields are parallel to the surface.

Find the enclosed charge Q_{enc} and apply Gauss's law:

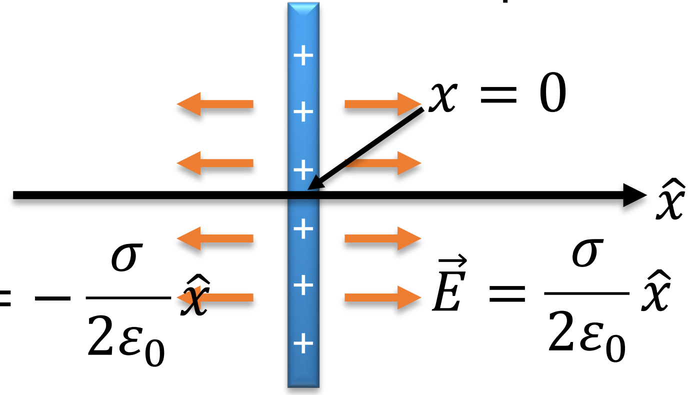
$$\oiint \vec{E} \cdot d\vec{A} = 4\pi k Q_{enc} = \frac{Q_{enc}}{\varepsilon_0}$$

Ref:

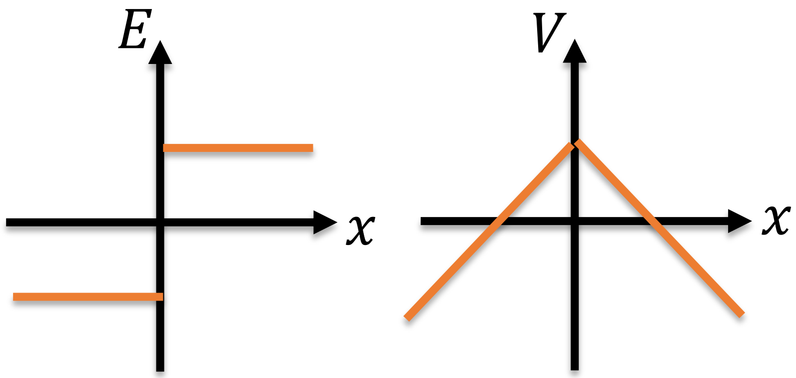
Discontinuity of Electric Fields

Electric Field & Electric Potential

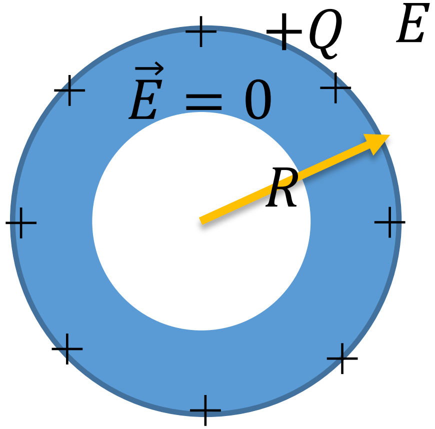
The electric field due to an infinite yz plane of positive charge at $x = 0$, the electric field in the space is



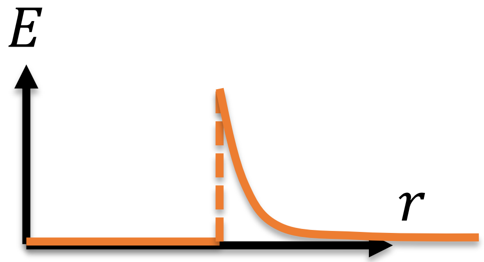
$$\vec{E} = -\frac{\sigma}{2\varepsilon_0}\hat{x} \qquad \vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{x}$$

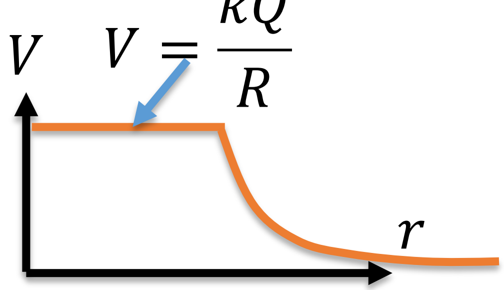


The electric field due to a charged metal sphere



$$\vec{E} = \frac{kQ}{r^2}\hat{r}$$





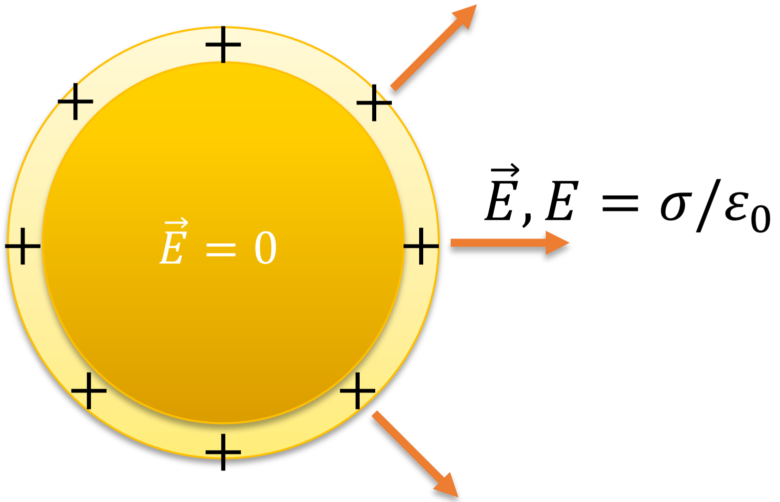
Ref:

Conductor

Inside a conductor, charges of the same polarity repel each other and they reside on surface of the conductor.

Charges redistribute themselves to give a zero electric field inside a conductor. (If the field is nonzero, the charges will move again.)

Electric field is perpendicular to the surface of conductors and the difference of electric field across the surface is $\Delta E = \sigma/\varepsilon_0$.



Ref:

Examples

An electric field is $\vec{E} = 100\hat{k}$ N/C ($z > 0$) and $\vec{E} = -100\hat{k}$ N/C ($z < 0$). A box of side length 1 m is placed with its center at the origin and its side planes parallel to xy, yz, and xz planes. (a) What is the flux enclosed by the surface? (b) What is the net charge in the box?

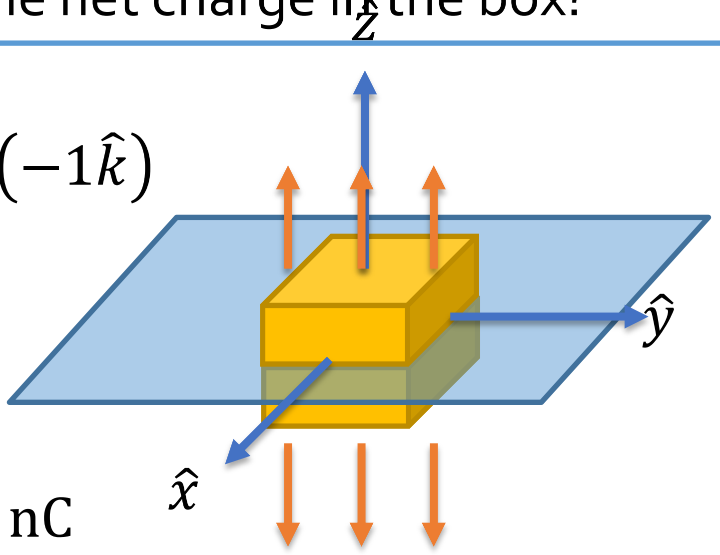
(a)

$$\Phi_{net} = \left(100\hat{k}\right)\cdot\left(1\hat{k}\right) + \left(-100\hat{k}\right)\cdot\left(-1\hat{k}\right)$$

$$\Phi_{net} = 200\ \mathrm{Nm^2/C}$$

(b)

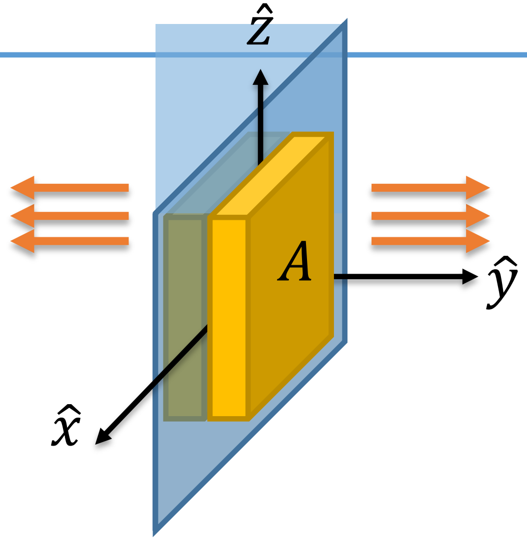
$$Q = \frac{\Phi_{net}}{4\pi k} = \frac{200}{4\pi\times 8.99\times 10^9} = 1.77\ \mathrm{nC}$$



Plane Symmetry

Examples

An infinite metal plate is charged with a charge density of $+\sigma$ and it is placed on the xz plane. Please calculate its electric field by using Gauss's law.



$$\oiint \vec{E} \cdot d\vec{A} = AE + AE + 0 = 4\pi k A\sigma$$

$$E = 2\pi k\sigma = \frac{1}{2}\frac{\sigma}{\varepsilon_0}$$

Ref:

Plane Symmetry

Examples

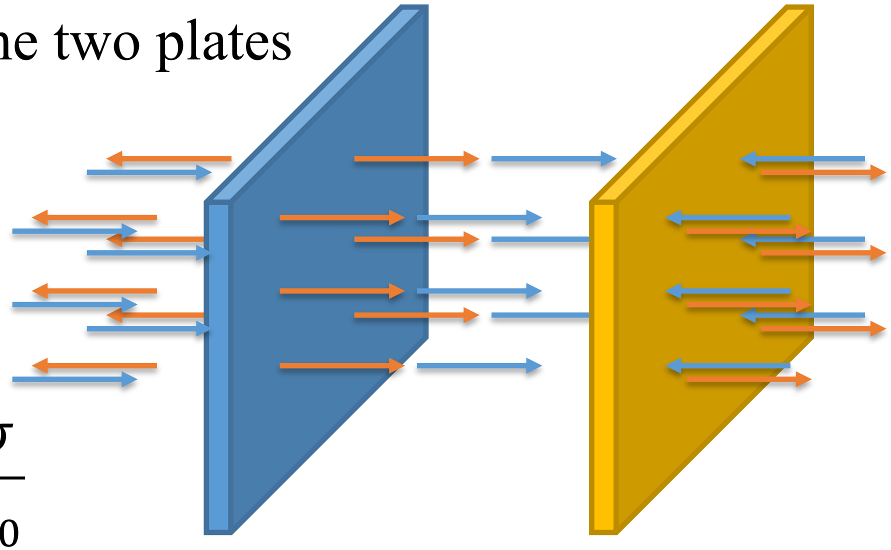
Two infinite metal plates are placed in parallel and charged with charge densities of $+\sigma$ and $-\sigma$. Please calculate the electric field inside the two plates.

Use superposition rule, outside the two plates

$$E = 0$$

inside the plates

$$E = 2\pi k\sigma + 2\pi k\sigma = 4\pi k\sigma = \frac{\sigma}{\varepsilon_0}$$



Ref:

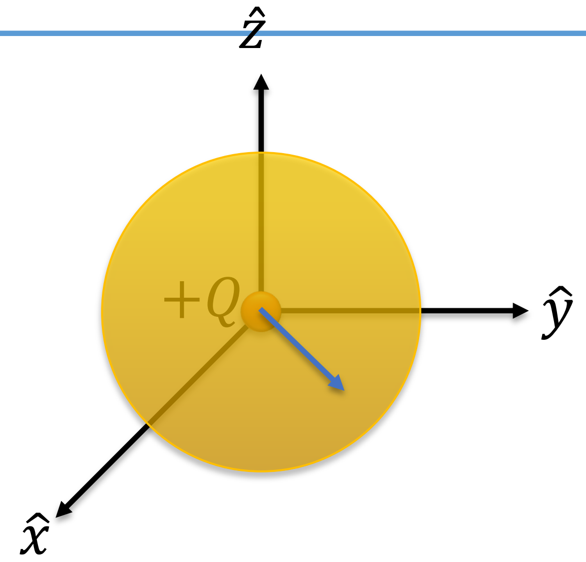
Spherical Symmetry

Examples

Use Gauss's law to find electric field of a point charge placed at the origin.

$$4\pi r^2 E = 4\pi k Q$$

$$E = \frac{kQ}{r^2} = \frac{Q}{4\pi\varepsilon_0 r^2}$$



Ref:

Spherical Symmetry, Shell

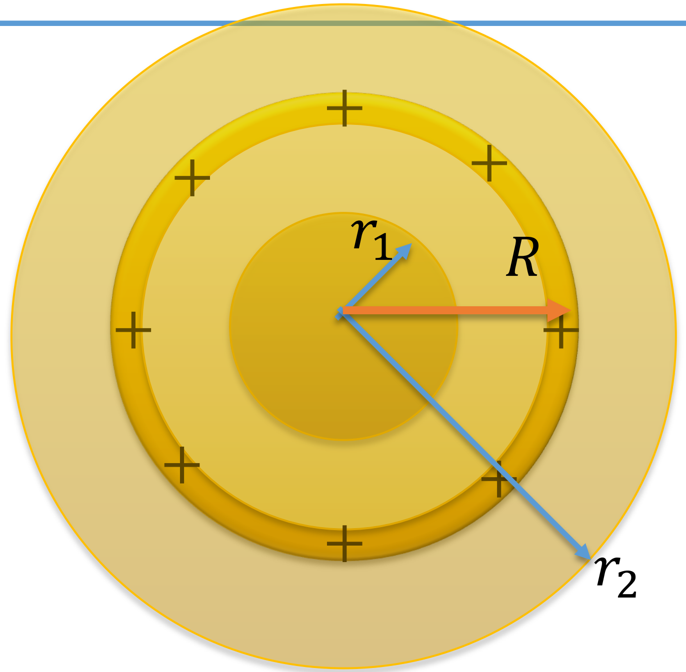
Examples

Use Gauss's law to find electric field inside and outside the charged conducting shell of charge Q and radius R.

$$r_1 < R \rightarrow 4\pi r_1^2 E = 0 \rightarrow E = 0$$

$$r_2 > R \rightarrow 4\pi r_2^2 E = 4\pi k Q$$

$$E = \frac{kQ}{r_2^2} = \frac{Q}{4\pi\varepsilon_0 r_2^2}$$

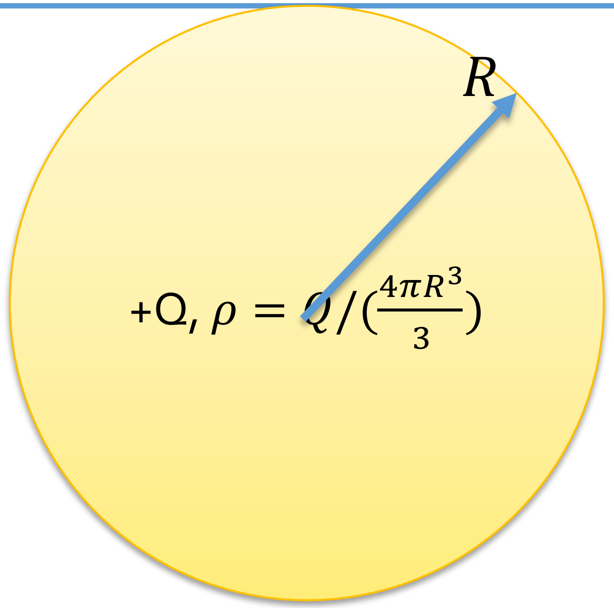


Ref:

Spherical Symmetry, Solid Sphere

Examples

Use Gauss's law to find electric field inside and outside the uniformly charged solid sphere of radius R and charge Q.



$r < R$

$$4\pi r^2 E = 4\pi k\left(\frac{4\pi}{3}r^3\rho\right)$$

$$E = \frac{4\pi k\rho r}{3} = \frac{\rho r}{3\varepsilon_0} = \frac{kQ}{R^2}\frac{r}{R} = \frac{Qr}{4\pi\varepsilon_0 R^3}$$

$r > R$

$$E = \frac{kQ}{r^2} = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Ref:

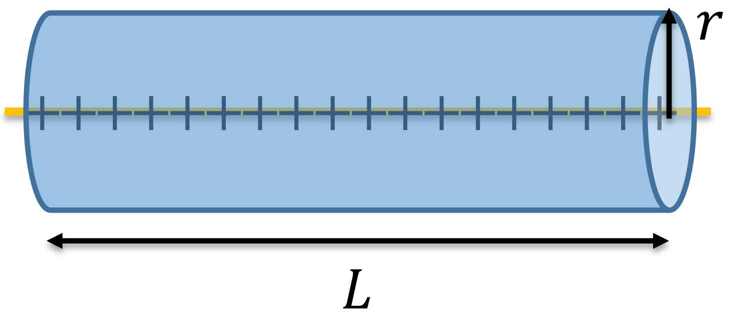
Cylindrical Symmetry, Line Charge

Examples

Please use Gauss's law to find the electric field of a charged infinite-long line. The density of the line charge is λ.

$$2\pi r L E = 4\pi k(\lambda L)$$

$$E = \frac{2k\lambda}{r} = \frac{\lambda}{2\pi\varepsilon_0 r}$$



Ref:

A sphere of radius R surrounds a particle with a charge Q, located at its center. Find the electric flux through a circular cap of half-angle θ_0.

Challenging problems

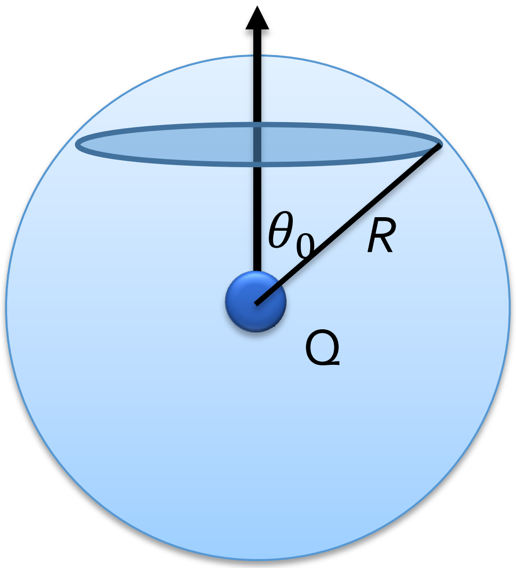
$$E = \frac{kQ}{r^2} \qquad \Phi_E = \int \vec{E} \cdot d\vec{A}$$

$$\Phi_E = \int \left(\frac{kQ}{r^2}\hat{r}\right) \cdot (dA\hat{r})$$

$$dA = (rd\theta)(r \sin\theta \, d\phi)$$

$$\Phi_E = \int_0^{\theta_0} \int_0^{2\pi} \frac{kQ}{R^2} R^2 \sin\theta \, d\phi d\theta$$

$$\Phi_E = \int_0^{\theta_0} \int_0^{2\pi} kQ \sin\theta \, d\phi d\theta$$



Challenging problems

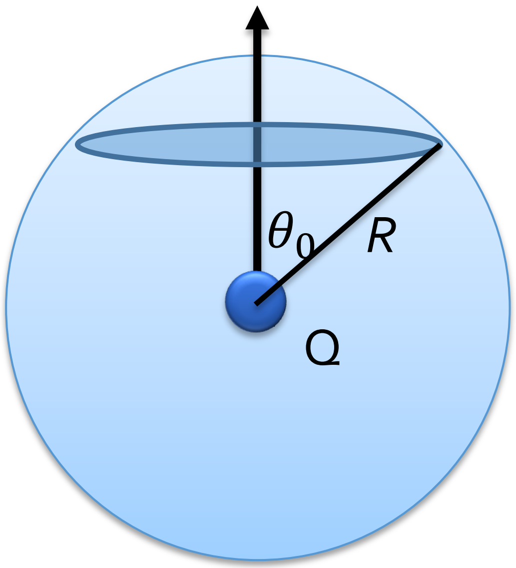
A sphere of radius R surrounds a particle with a charge Q, located at its center. Find the electric flux through a circular cap of half-angle θ_0.

$$\Phi_E = \int_0^{\theta_0} 2\pi k Q \sin\theta \, d\theta$$

$$\Phi_E = 2\pi k Q [-\cos\theta]_0^{\theta_0}$$

$$\Phi_E = 2\pi k Q(-\cos\theta_0 + \cos(0))$$

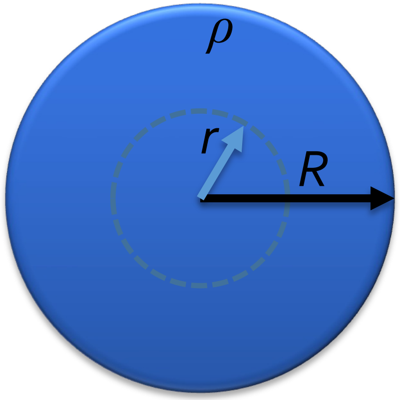
$$\Phi_E = 2\pi k Q(1 - \cos\theta_0) = \frac{Q}{2\varepsilon_0}(1 - \cos\theta_0)$$

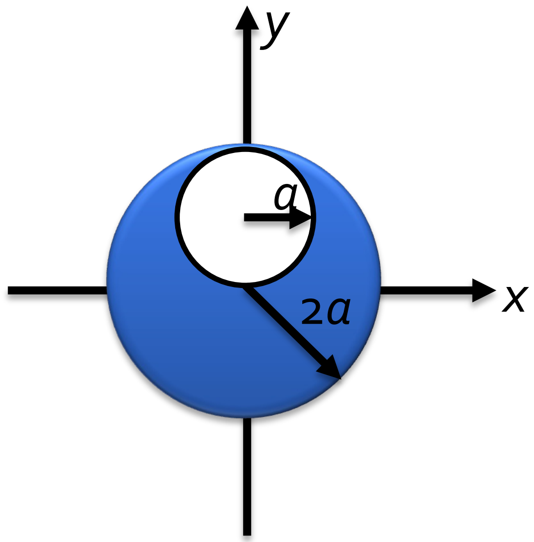


A sphere of radius $2a$ is made of a non-conducting material that has a uniform volume charge density ρ. Assume the material does not affect the electric field. A spherical cavity of radius a is now removed from the sphere as shown in the right figure. Show that the electric field within the cavity is uniform and its magnitude is given by $\rho a/3\varepsilon_0$.

Challenging problems

uniformly charged insulating sphere with volume charge density





$$4\pi r^2 E = 4\pi k q = \frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\rho\frac{4\pi r^3}{3}$$

$$E = \frac{\rho r}{3\varepsilon_0} \rightarrow \vec{E} = \frac{\rho r}{3\varepsilon_0}\hat{r}$$

Challenging problems

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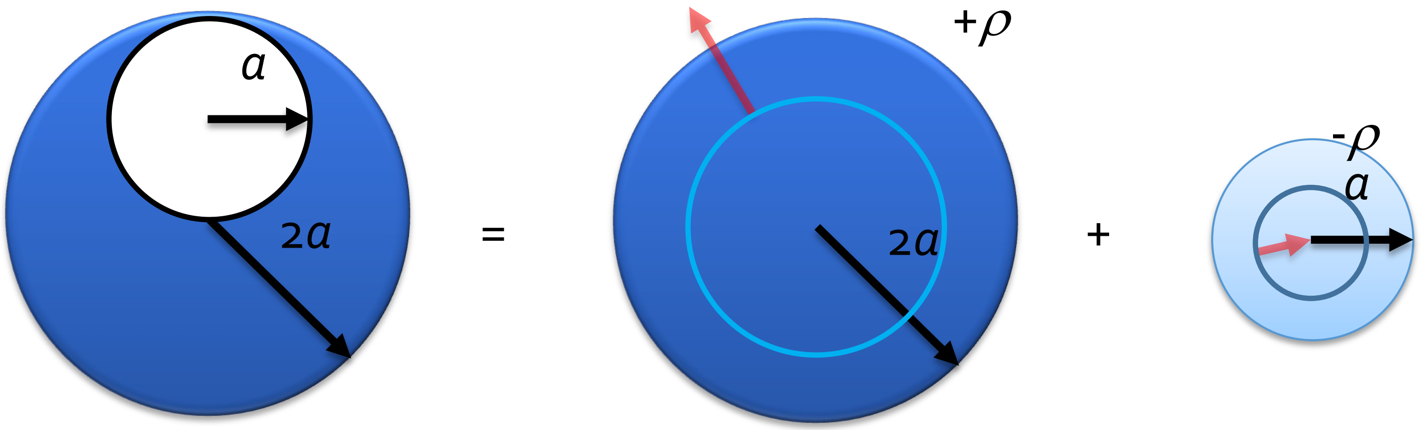

$$\vec{E}_{\rho+} = \frac{\rho r}{3\varepsilon_0}\hat{r} \qquad \vec{E}_{\rho-} = -\frac{\rho r}{3\varepsilon_0}\hat{r}$$

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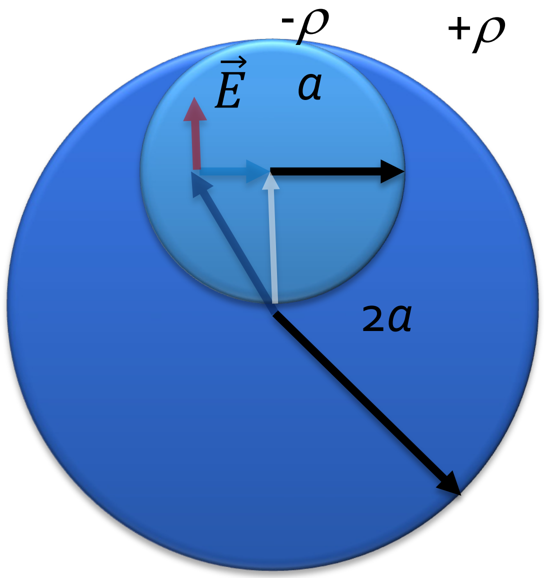
Challenging problems



$$E = \frac{\rho a}{3\varepsilon_0}$$