Chapter 25 Capacitance and Dielectrics

Work must be done when charging a conductor. The energy of the work is the same as the energy require to construct electric fields surround the conductor. Use the energy and electric field of a capacitor, we will derive the energy density of electric field.

25.1 Definition of Capacitance

25.2 Calculating Capacitance

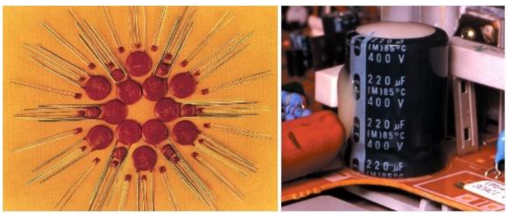
Measure of **the capacity to store charge**: $C = \dfrac{Q}{V}$

Unit: farad (F): 1 F = 1 C / V; 1 μF = 10^{-6} F; 1 pF = 10^{-12} F

The ratio of charge Q to the potential depends on the size and the shape of the conductor.

Capacitors

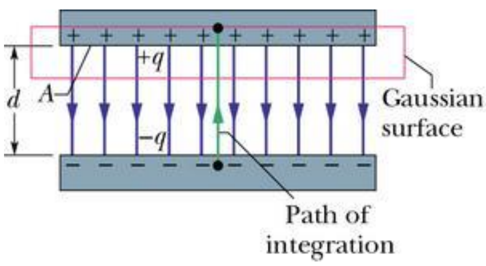
A device consisting of two conductors carrying equal but opposite charges is called a capacitor.



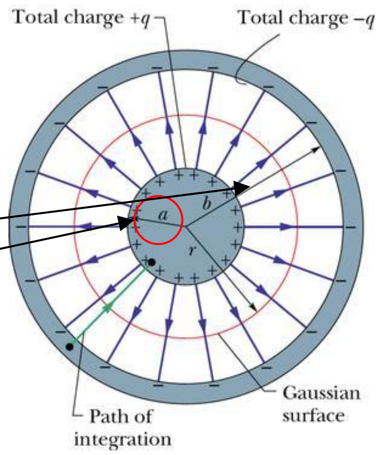
Parallel Plate Capacitor

$$E = 4\pi k\sigma, V = 4\pi k\sigma d = 4\pi k d Q/A$$

$$C = \frac{Q}{V} = \frac{A}{4\pi k d} = \frac{A\varepsilon_0}{d}$$



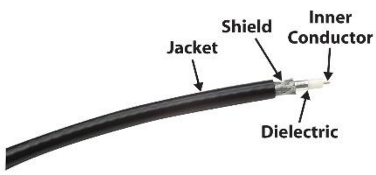
Cylindrical Capacitor



$$2\pi r L E = \frac{Q}{\varepsilon_0} \quad \text{-->} \quad E_R = \frac{Q}{2\pi r L \varepsilon_0}$$

$$V = V_a - V_b = -\int_b^a \frac{Q}{2\pi r L \varepsilon_0} dr = \frac{Q}{2\pi \varepsilon_0 L} \ln\left(\frac{b}{a}\right)$$

$$C = \frac{Q}{V} = \frac{2\pi \varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}$$



Spherical Capacitor

$$4\pi r^2 E = \frac{Q}{\varepsilon_0}, \quad V = -\int_b^a \frac{Q}{4\pi \varepsilon_0 r^2} dr = \frac{Q}{4\pi \varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$C = \frac{Q}{V} = 4\pi \varepsilon_0 \frac{ab}{b-a}$$

Self-Capacitance

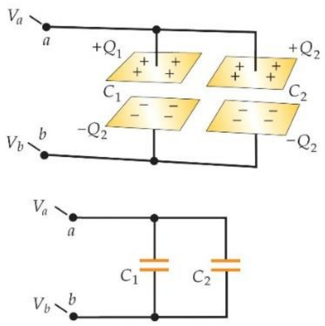
The potential of a spherical conductor of radius R carrying a charge Q is $V = \dfrac{kQ}{R}$.

The self-capacitance of a spherical conductor is:

$$C = \frac{Q}{V} = \frac{Q}{\frac{kQ}{R}} = \frac{R}{k} = 4\pi \varepsilon_0 R$$

25.3 Combination of Capacitors

Capacitors Connected in Parallel



Obtain V and Q to calculate C.

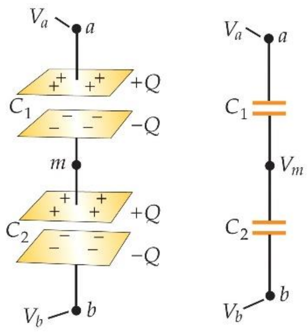
$V_1 = V_2 = V \quad \& \quad C_1 = Q_1 / V_1$

$$Q = Q_1 + Q_2 = C_1V_1 + C_2V_2 = (C_1 + C_2)V$$
$$C = Q/V = C_1 + C_2$$

Capacitors connected in parallel:

$$C_{eq} = C_1 + C_2 + C_3 + C_4 + \ldots$$

Capacitors connected in series



$$Q_1 = Q_2 = Q \quad \& \quad C_1 = Q_1/V_1$$

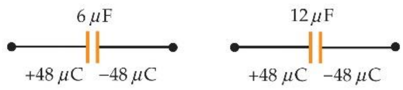
$$V = V_1 + V_2 = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

$$C = \frac{Q}{V} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

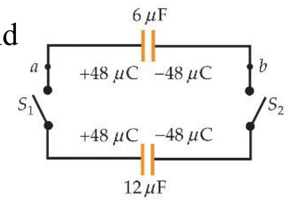
series --> sum voltage & parallel --> sum charges

Capacitors connected in series:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \frac{1}{C_4} + \ldots$$



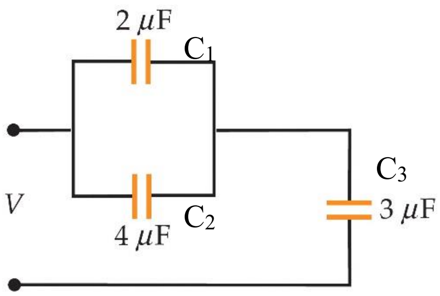
Example: Two capacitors are removed from the battery and carefully connected from each other.



$$V_1 = V_2 \quad \& \quad C_1 = \frac{Q_1}{V_1}$$

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2} \quad \& \quad Q_1 + Q_2 = 96\mu C$$

Capacitors in Series and in Paral



$$\frac{1}{C} = \frac{1}{C_1 + C_2} + \frac{1}{C_3}$$

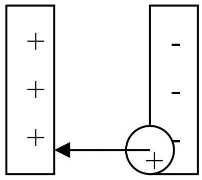
Q = ? V = ?

$$Q = Q_3 = Q_1 + Q_2, \quad V = V_1 + V_3, \quad V_1 = V_2$$

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2} \quad \& \quad Q_1 + Q_2 = Q \quad \text{-->} \quad Q_1 = \frac{C_1}{C_1 + C_2}Q \quad \& \quad Q_2 = \frac{C_2}{C_1 + C_2}Q$$

$$V = V_1 + V_3 = \frac{Q_1}{C_1} + \frac{Q_3}{C_3} = \frac{1}{C_1 + C_2} Q + \frac{1}{C_3} Q \quad \text{-->} \quad C = \frac{Q}{V} = \frac{1}{\frac{1}{C_1 + C_2} + \frac{1}{C_3}}$$

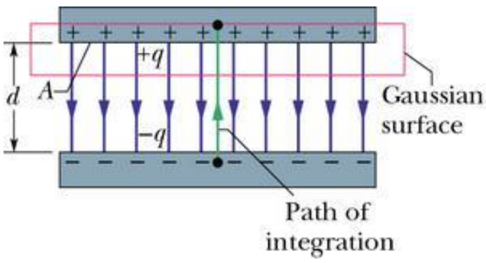
25.4 Energy Stored in a Charged Capacitor



$$dU = Vdq = \frac{q}{C} dq$$

$$U = \int dU = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C} = \frac{1}{2} QV = \frac{1}{2} CV^2$$

Electrostatic Field Energy (derived from energy stored in a capacitor)

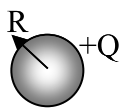


$$C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}, \quad V = Ed$$

$$U = \frac{1}{2} CV^2 = \frac{1}{2} \frac{\varepsilon_0 A}{d} (Ed)^2 = \left(\frac{1}{2} \varepsilon_0 E^2\right) Ad = \left(\frac{1}{2} \varepsilon_0 E^2\right) V$$

Electrostatic Energy Density: $u_e = \frac{U}{V} = \frac{1}{2} \varepsilon_0 E^2$ (energy per unit volume)

Example: Calculate the energy stored in the conductor carrying a charge Q.



$r < R: \quad E = 0$

$r > R: \quad E = \frac{kQ}{r^2}$

$$dU = u_e dV = \left(\frac{1}{2} \varepsilon_0 E^2\right)(4\pi r^2 dr) = \left(\frac{1}{2} \varepsilon_0 \frac{k^2 Q^2}{r^4}\right)(4\pi r^2 dr)$$

$$U = \int_R^\infty dU = 2\pi \varepsilon_0 k^2 Q^2 \int_R^\infty \frac{1}{r^2} dr = \frac{1}{2} Q \frac{kQ}{R} = \frac{1}{2} QV$$

25.5 Capacitors and Dielectrics

When the space between the two conductors of a capacitor is occupied by a dielectric, the capacitance is increased by a factor κ $(\kappa > 1)$ that is characteristic of the dielectric.

If the dielectric field is E_0 before the dielectric slab is inserted, after the dielectric slab is inserted between the plates the field is

$E = \frac{E_0}{\kappa}$ --> the potential is $V = Ed = \frac{E_0 d}{\kappa} = \frac{V_0}{\kappa}$

If $C_0 = \frac{Q}{V_0}$, the capacitor is $C' = \frac{Q}{V} = \frac{Q}{V_0/\kappa} = \kappa C_0$.

The capacitance of a parallel-plate capacitor filled with a dielectric of constant κ is

$C = \frac{Q}{V} = \frac{A\sigma}{V_0/\kappa} = \frac{A\sigma}{E_0 d}\kappa = \frac{A\sigma}{\frac{\sigma}{\varepsilon_0}d}\kappa = \frac{A\kappa\varepsilon_0}{d} = \frac{A\varepsilon}{d}$ --> $\varepsilon = \kappa\varepsilon_0$ is called the permittivity of the dielectric.

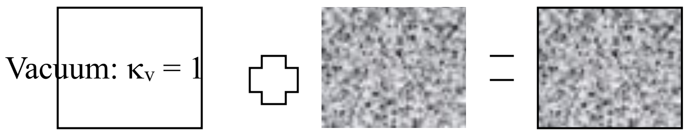


TABLE 24-1

Dielectric Constants and Dielectric Strengths of Various Materials

Material	Dielectric Constant κ	Dielectric Strength, kV/mm
Air	1.00059	3
Bakelite	4.9	24
Glass (Pyrex)	5.6	14
Mica	5.4	10–100
Neoprene	6.9	12
Paper	3.7	16
Paraffin	2.1–2.5	10
Plexiglas	3.4	40
Polystyrene	2.55	24
Porcelain	7	5.7
Transformer oil	2.24	12

Energy Stored in The Presence of a Dielectric

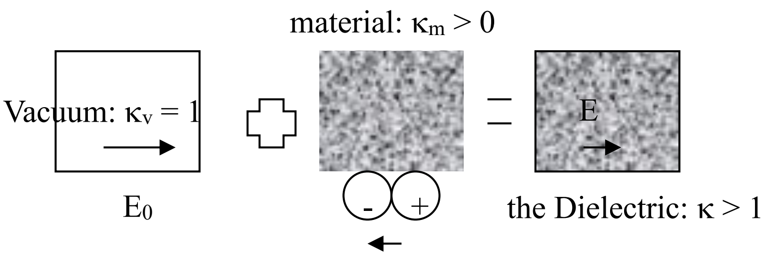
The energy stored in a capacitor is:

$$dU = Vdq \text{ --> } dU = \frac{q}{C}dq \text{ --> } \int_0^U dU = \int_0^Q \frac{q}{C}dq \text{ --> } U = \frac{Q^2}{2C} = \frac{1}{2}CV^2$$

The energy of a capacitor with the dielectric is

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{A\varepsilon}{d}(Ed)^2 = \frac{1}{2}\frac{A\varepsilon}{d}(Ed)^2 = \left(\frac{1}{2}\varepsilon E^2\right)(Ad)$$

$$u_e = \frac{1}{2}\varepsilon E^2 = \frac{1}{2}\kappa\varepsilon_0 E^2$$



1. You lose electric force to separate the charge.
2. You enlarge the charging capacity as you know the dielectric will breakdown in a high electric field. (If the same charge --> you lose some electric field,)
3. You increase the energy per unit volume.

Combination of Capacitors

Example: A parallel-plate capacitor has square plates of edge length 10 cm and a separation of d = 4 mm. A dielectric slab of constant $\kappa = 2$ has dimensions 10 cm X 10 cm X 4 mm. (a) What is the capacitance without the dielectric? (b) What is the capacitance with the dielectric? (c) What is the capacitance if a dielectric slab with dimensions 10 cm X 10 cm X 3 mm is inserted into the 4-mm gap?

(a) $C = \frac{A\varepsilon_0}{d}$, (b) $C = \frac{A\varepsilon}{d}$

(c) series connection: $\frac{1}{C} = \frac{1}{\frac{A\varepsilon_0}{d/4}} + \frac{1}{\frac{A\varepsilon}{3d/4}} = \frac{1}{\frac{A\varepsilon_0}{d/4}} + \frac{1}{\frac{A\kappa\varepsilon_0}{3d/4}}$

Example: The parallel plates of a given capacitor are square with $A = a^2$ and separation distance d. If the plates are maintained at a constant potential V and a square of dielectric slab of constant κ, area $A = a^2$, thickness d is inserted between the capacitor plates to a distance x as shown in the following figure. Let σ_0 be the free charge density at the conductor-air surface. (a) Calculate the free charge density σ_κ at the capacitor-dielectric surface. (b) What is the effective capacitance? (c) What is the magnitude of the required force to prevent the dielectric slab from sliding into the plates?

(a) In air: $E = \dfrac{\sigma_0}{\varepsilon_0}$ & $V = \dfrac{\sigma_0}{\varepsilon_0} d$, in dielectric: $E = \dfrac{\sigma_K}{K\varepsilon_0}$ & $V = \dfrac{\sigma_K}{K\varepsilon_0} d$

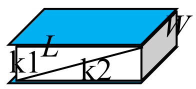
$\rightarrow$ $\sigma_K = K\sigma_0$

(b) $C = C_1 + C_2 = \dfrac{axK\varepsilon_0}{d} + \dfrac{a(a-x)\varepsilon_0}{d} = \dfrac{a\varepsilon_0}{d}\left(a + (K-1)x\right)$

(c) $U = \dfrac{1}{2}CV^2$ $\rightarrow$

$$F = -\left(\frac{1}{2}V^2\right)\frac{dC}{dx} + V\frac{dQ}{dx} = -\left(\frac{1}{2}V^2\right)\frac{dC}{dx} + V^2\frac{dC}{dx} = \frac{1}{2}V^2\frac{a\varepsilon_0}{d}(K-1)$$

The first term is due to charge redistribution and the second is due to the additional charges supplied by the constant voltage.



Example: A parallel plate capacitor with plates of area LW and separation t has the region between its plates filled with wedges of two dielectric materials. Assume t is much less than both W and L. (a) Please determine its capacitance.

The thickness of the k1 material decrease as a function of $\dfrac{t(L-x)}{L}$ while that of the k2 material is of $\dfrac{tx}{L}$

For a short stripe of dx, $C_1 = \dfrac{A_1\varepsilon}{d_1} = \dfrac{Wdx(\kappa_1\varepsilon_0)}{t(L-x)/L}$, $C_2 = \dfrac{A_1\varepsilon}{d_1} = \dfrac{Wdx(\kappa_2\varepsilon_0)}{tx/L}$

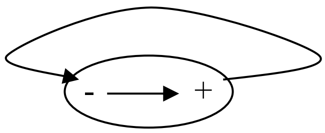
The series connected capacitance $\dfrac{1}{C} = \dfrac{1}{C_1} + \dfrac{1}{C_2}$, $C = \dfrac{W\varepsilon_0 dx}{t\left(\dfrac{1}{\kappa_1} + \left(\dfrac{1}{\kappa_2} - \dfrac{1}{\kappa_1}\right)\dfrac{x}{L}\right)}$

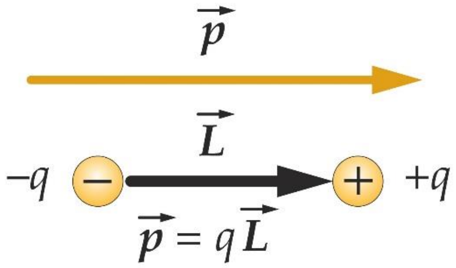
The total parallel connected capacitance

$$C_{total} = \int_0^L \frac{W\varepsilon_0 dx}{t\left(\dfrac{1}{\kappa_1} + \left(\dfrac{1}{\kappa_2} - \dfrac{1}{\kappa_1}\right)\dfrac{x}{L}\right)} = \frac{W\varepsilon_0 L}{t\left(\dfrac{1}{\kappa_2} - \dfrac{1}{\kappa_1}\right)}\ln\left(\frac{\kappa_1}{\kappa_2}\right)$$

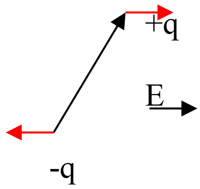
25.6 Electric Dipole in an Electric Field

Inside the material → to make sure that the electric field lines are from the positive charge to the negative charge





If the field is uniform, it can rotate the dipole.



$$\vec{\tau} = \vec{r}_+ \times q\vec{E} + \vec{r}_- \times \left(-q\vec{E}\right) = q\vec{d} \times \vec{E} = \vec{p} \times \vec{E}$$

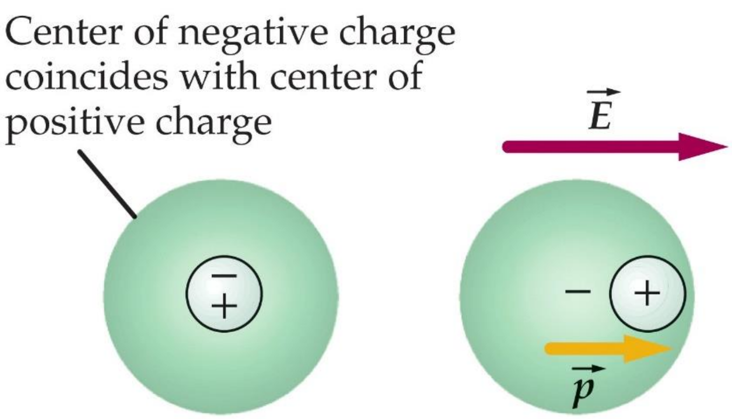
$$\vec{\tau} = \vec{p} \times \vec{E} = pE\sin\theta$$

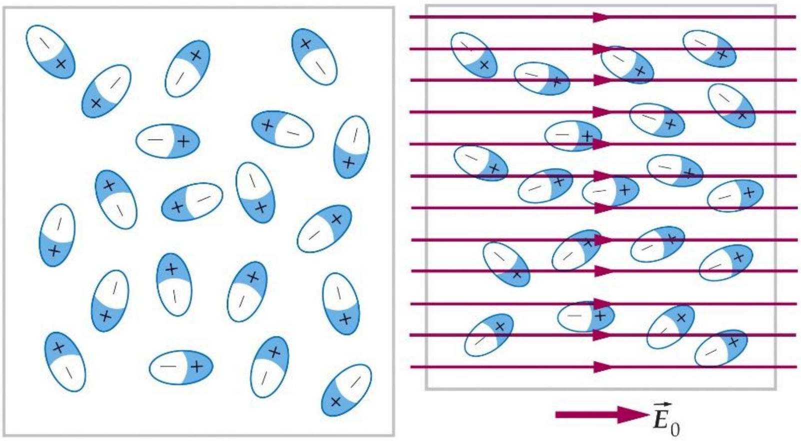
The potential energy: $dU = -Fdx = -\tau d\theta = -(-pE\sin\theta)d\theta$

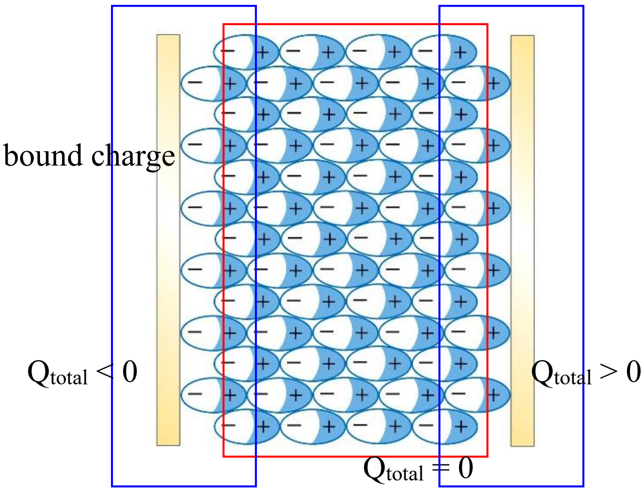
$$\rightarrow \; U = pE\int_{\theta i}^{\theta f} \sin\theta d\theta = -pE(\cos\theta_f - \cos\theta_i)$$

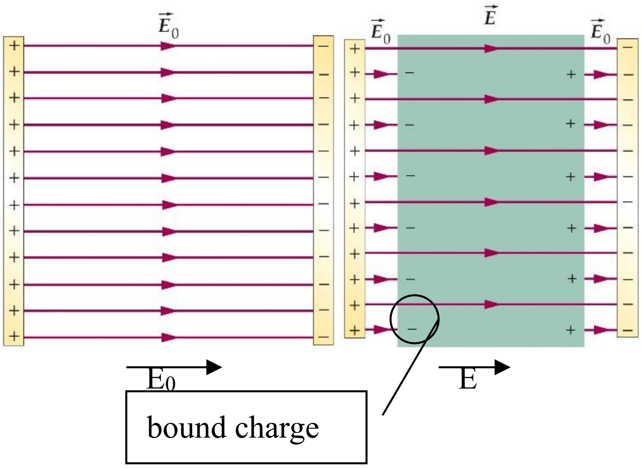
$$U = -\vec{p} \cdot \vec{E}$$

25.7 An Atomic Description of Dielectrics







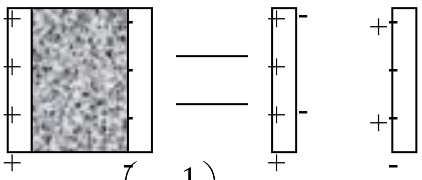


Example: A hydrogen atom consists of a proton nucleus of charge +e and an electron of charge –e. The charge distribution of the atom is spherically symmetric, so the atom is nonpolar. Consider a model in which the hydrogen atom consists of a positive charge +e at the center of a uniformly charged spherical cloud of radius R and total charge –e. Show that when such an atom is placed in a uniform external field $\vec{E}$, the induced dipole moment is proportional to $\vec{E}$; that is, $\vec{p} = \alpha\vec{E}$, where α is called the polarizability.

$$E_{inside\_from\_-e} = \frac{Q}{A\varepsilon_0} = \frac{\frac{4\pi}{3} r^3 \frac{-e}{\frac{4\pi R^3}{3}}}{4\pi r^2 \varepsilon_0} = \frac{-e}{4\pi\varepsilon_0 R^3} r$$

$$p = er = \alpha E = \alpha \frac{er}{4\pi\varepsilon_0 R^3} \text{ --> } \alpha = 4\pi\varepsilon_0 R^3$$

Magnitude of The Bound Charge



$$E_b = \frac{\sigma_b}{\varepsilon_0} \quad \& \quad E_0 = \frac{\sigma_f}{\varepsilon_0}$$

$$E = \frac{E_0}{\kappa} = E_0 - E_b \text{ --> } E_b = \left(1 - \frac{1}{\kappa}\right) E_0 \text{ --> } \sigma_b = \left(1 - \frac{1}{\kappa}\right) \sigma_f$$

$$\sigma_{effective} = \sigma_f - \sigma_b = \frac{1}{\kappa} \sigma_f = \sigma_0 \rightarrow \sigma_f = \kappa \sigma_0$$