Lecture 11 Conservation of Angular Momentum

11.1 The Vector Product and Torque

The Cross Product:

$\vec{\tau} = \vec{r} \times \vec{F}$

Define the cross product: $\hat{i} \times \hat{i} = 0$, $\hat{i} \times \hat{j} = \hat{k}$, … → $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

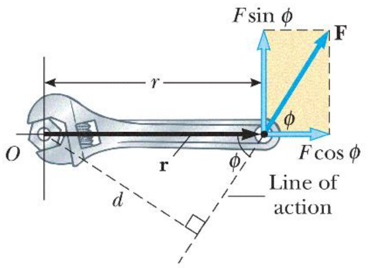
$\vec{A} \times \vec{B} = |A||B| \sin\theta$

Rules: $\vec{A} \times \vec{A} = 0$, $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$, $\frac{d}{dt}\left(\vec{A} \times \vec{B}\right) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$

Serway/Jewett; Principles of Physics, 3/e
Figure 10.11

11.2 Angular Momentum: The Nonisolated System



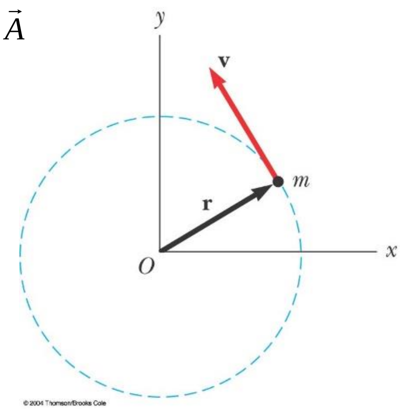
$\vec{\tau} = \vec{r} \times \vec{F}$

Define a torque about an axis

Example: The vector product

Two vectors lying in the xy plane are given by the equations $\vec{A} = 2 \cdot \hat{i} + 3 \cdot \hat{j}$, $\vec{B} = -\hat{i} + \hat{j}$.

Find $\vec{A} \times \vec{B}$, and verify explicitly that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$



$\vec{L} = \vec{r} \times \vec{P}$

Define the momentum about an point

$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{P}}{dt} + \frac{d\vec{r}}{dt} \times \vec{P} = \vec{r} \times \vec{F}$

$L = \sum m_i v_i r_i = \sum m_i r_i \omega \cdot r_i = \sum m_i r_i^2 \omega = I\omega$

A System of Particles:

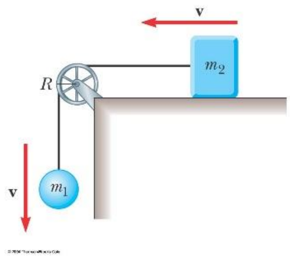
The net external torque acting on a system of particles gives

$$\vec{\tau}_{ext} = \frac{d\vec{L}_{system}}{dt} \rightarrow \Delta\vec{L}_{system} = \int \vec{\tau}_{ext} dt$$

Example: Angular momentum of a Particle in circular motion

A particle moves in the xy plane in a circular path of radius r. Find the magnitude and direction of its angular momentum relative to O when its linear velocity is v.

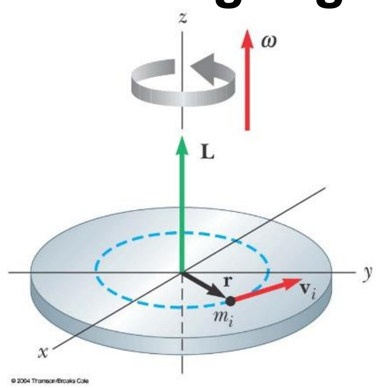
Example: A System of Objects



$$L = m_1 vR + MR^2\omega + m_2 vR = m_1 vR + MRv + m_2 vR$$

$$\frac{dL}{dt} = \tau = R(mg) \rightarrow mgR = (m_1 + m_2 + M)R\frac{dv}{dt}$$

11.3 Angular Momentum of a Rotating Rigid Object



$$L_i = m_i r_i^2 \omega$$

$$L_z = \sum_i L_i = \sum_i m_i r_i^2 \omega = \left(\sum_i m_i r_i^2\right)\omega = I\omega$$

$$\sum \tau_{ext} = \frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha$$

Example: Bowling Ball

Estimate the magnitude of the angular momentum of a bowling ball spinning at 10 rev/s. A typical bowling ball might have a mass of 6.00 kg and a radius of 12 cm.

$$I_{ball} = \frac{2}{5}MR^2 = \frac{2}{5}(6.0)(0.12)^2 = 0.035 kg\cdot m^2$$

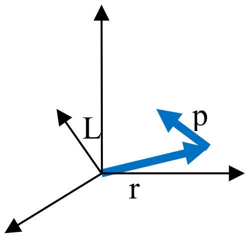
$$L_z = I\omega = (0.035)(10\cdot 2\pi) = 2.2 kg\cdot m^2/s$$

11.4 The Isolated System: Conservation of

Angular Momentum

What is the total angular momentum? The earth's orbital system gives an example.

$$\vec{L}_{system} = \vec{L}_{orbit} + \vec{L}_{spin}$$



$$\vec{L}_{orbit} = \vec{r} \times \vec{p} = \vec{r} \times (M\vec{v})$$

$$\vec{L} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k} \quad (L_z = \vec{r}_{xy} \times \vec{p}_{xy} = xp_y - yp_x \quad \ldots)$$

$$\rightarrow \quad \tau_z = \frac{dL_z}{dt} = I_z\alpha$$

正如你的大腦，賽格威知道你正向前傾時，它會運用適當的速度轉動輪子，所以車子就會向前行，賽格威最基本的組合，是一系列的感應器，控制系統及馬達系統，在本段說明，我們來看一下每個部份的元件基本感應系統是一組陀螺儀組合，基本的陀螺儀有如置於穩定設計裝置的輪脊。

輪脊可感應旋轉軸感的變化，因為作用力會跟隨物體本身移動，如果你在輪脊的頂端某一點施力，當它仍感應到你的作用力時，這一點會移到輪子前端，當施力點繼續移動時，可經由輪子另一端的作用來結束作用力，以使本身可以自動平衡，因為可抗外部的力量，陀螺儀可凌空維持相對地面的位置，即使你將其傾斜，陀螺儀仍可凌空自由移動。經衡量陀螺儀輪脊相對於輪圈的位置，精細的感應器會告知物體的傾斜程度（傾斜程度相較離垂直位置有多遠）以及其傾斜率（傾斜速度有多快）。

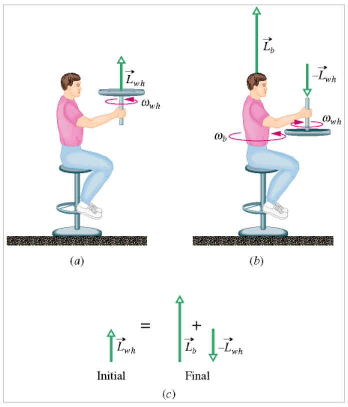
平衡感應組（BSA），來自矽感應系統公司，為一組設計優雅，極度耐用及不可置信的靈敏感應器。此一小型立方體，每邊 3 英吋，內含五個立體狀震環式角度測量的感應器（陀螺儀），運用科氏力效應來衡量旋轉速度，這些小環型物為電機震動，當轉動時會偵測到微力的產生。

每個〝陀螺儀〞以獨特的角度安置著，可用來衡量多個方向，開機時，賽格威的機板會持續比對，來判定五個陀螺儀所提供的資料是否有誤，如果判定有任何的陀螺儀有失誤，其他的陀螺儀可以用來補償校正錯誤的資料直到車子安全關機停車。另有兩個內裝有電解液的水平傾斜儀，可以提供有如人類內耳中的重力平衡資訊，平衡感應組經由二個獨立的微處理器所監控，並平均分成二組安全備援系統。甚至二邊亦是經由光學傳導以避免電子失誤傳到另一邊。

除了所有 Segway HT 型號都具備的安全特性之外，i 系列另具備許多的標準功能：動力輔助模式(Power assist mode) 可以讓使用者上下樓梯移動自如 ，控制軸可以調成適合騎乘的高度以及放到汽車的後車廂，另外為了避免被偷，每台 Segway HT 配有專屬的 64 位元加密認證鑰匙， i 系列亦配備機械式停車腳架。

When the net external torque acting on a system remains zero, we have

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt} = 0 \text{ or } \vec{L} = const .$$



Sample Problem:

Figure 12-19*a* shows a student, again sitting on a stool that can rotate freely about a vertical axis. The student, initially at rest, is holding a bicycle wheel whose rim is loaded with lead and whose rotational inertia I_{wh} about its central axis is 1.2 kg·m^2. The wheel is rotating at an angular speed ω_{wh} of 3.9 rev/s; as seen from overhead, the rotation is counterclockwise. The axis of the wheel is vertical, and the angular momentum $\vec{L}_{wh}$ of the wheel points vertically upward. The student now inverts the wheel (Fig. 12-19*b*) so that, as seen from overhead, it is rotating clockwise. Its angular momentum is then $-\vec{L}_{wh}$. The inversion results in the student, the stool, and the wheel's center rotating together as a composite rigid body about the stool's rotation axis, with rotational inertia I_b = 6.8 kg·m^2. (The fact that the wheel is also rotating about its center does not affect the mass distribution of this composite body; thus, I_b has the same value whether or not the wheel rotates.) With what angular speed ω_b and in what direction does the composite body rotate after the inversion of the wheel?

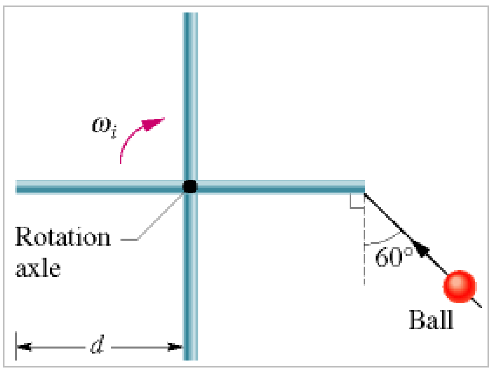
$$L_{b,i} + L_{wh,i} = L_{b,f} + L_{wh,f} \text{ , } L_{b,i} = 0 \text{ , } L_{wh,i} = I_{wh} \cdot \omega_{wh} = 4.68$$

$$L_{b,f} = 2L_{wh} = 9.36$$

$$\omega = \frac{L_{b,f}}{I_b} = 1.4 rev/s$$

Sample Problem:

In the overhead view of Fig. 12-21 , four thin, uniform rods, each of mass M and length d = 0.50 m, are rigidly connected to a vertical axle to form a turnstile. The turnstile rotates clockwise about the axle, which is attached to a floor, with initial angular velocity ω_i = -2.0 rad/s. A mud ball of mass $m = \frac{1}{3} M$ and initial speed v_i = 12 m/s is thrown

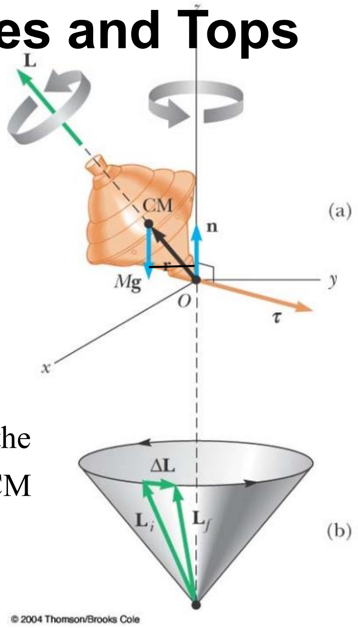


along the path shown and sticks to the end of one rod. What is the final angular velocity ω_f of the ball–turnstile system?

$$I_i = \frac{1}{12}2M \cdot (2d)^2 \cdot 2 = \frac{4}{3}Md^2, \quad L_i = I_i\omega_i - d \cdot mv_i \cdot \cos 60^o = \frac{4}{3}Md^2\omega_i - \frac{1}{2}d\frac{1}{3}Mv_i$$

$$I_f = \frac{4}{3}Md^2 + \frac{M}{3}d^2, \quad I_f\omega_f = L_i, \quad \omega_f = \frac{\frac{4}{3}Md^2\omega_i - \frac{1}{6}dMv_i}{\frac{4}{3}Md^2 + \frac{1}{3}Md^2} = -0.8$$

11.5 The Motion of Gyroscopes and Tops



What is rotation? What is precessional motion?
A Top is spinning about its body axis and the body axis moves about the z-axis.

The top rotates about its symmetry axis (principal axis).

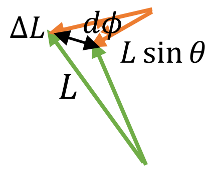
Its angular momentum is $\vec{L} = I_{about\_the\_principal\_axis}\vec{\omega}$

The gravitational force may give a torque to change the direction of the angular momentum. Let the position of CM be $\vec{R}$ from the origin O.

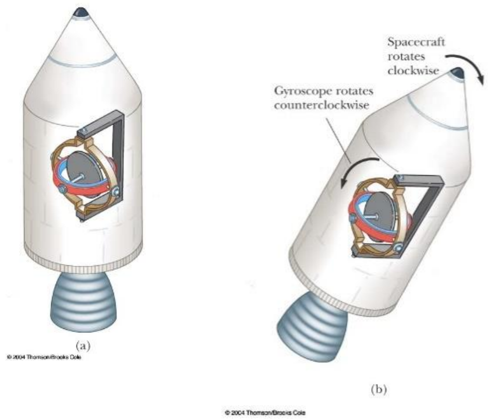
$$\tau = RMg\sin\theta$$

$$d\phi = \frac{dL}{L\sin\theta}, dL = \tau dt = RMg\sin\theta\, dt$$

$$d\phi = \frac{RMg\sin\theta\, dt}{L\sin\theta} \rightarrow \omega_P = \frac{d\phi}{dt} = \frac{RMg}{L} = \frac{RMg}{I\omega}$$



Here ω_P is the precessional frequency of the top.



The motion of gyroscope and the conservation of angular momentum are applied in finding the direction for space travel.

Example: A particle of mass m moves with speed v_0 in a circle of radius r_0 on a frictionless tabletop. The particle is attached to a string that passes through a hole in the table. The string is slowly pulled downward so that the particle moves in a small circle of radius r_1. (a) Find final velocity in terms of v_0, r_0, r_1. (b) Find the tension when the particle is moving in a circle of radius r in terms of m, r, and the angular momentum $\vec{L}$. (c) Calculate the work done on the particle by the tension $\vec{T}$ by integrating $\int \vec{T} \cdot d\vec{l}$. Express your answer in terms of r and L_0.

(a) $L_0 = r_0 m v_0 = r_1 m v_1 \rightarrow v_1 = r_0 m_0 / r_1$

(b) $T = m a_r = m \dfrac{v^2}{r} = \dfrac{L^2}{m r^3}$

(c) $-\int \dfrac{L^2}{m r^3} dr = \dfrac{L^2}{2m}\left(\dfrac{1}{r^2} - \dfrac{1}{r_0^2}\right)$

Quantization of Angular Momentum

It has been found that the angular momentum of a system is a fundamental quantity. To explain the experimental results, we rely on the fact that angular momentum has discrete values which are multiples of the fundamental unit of angular momentum:

$\hbar = \dfrac{h}{2\pi} = 1.054 \times 10^{-34} kg \cdot m^2 / s$.

$L = mvr = n\hbar$

Let us accept the postulate, consider the O_2 molecule, we can find the order of magnitude of the lowest angular speed

$$I_{CM}\omega \sim \hbar, \quad \omega \sim \frac{1.054 \times 10^{-34}}{2 \cdot \left(2.66 \times 10^{-26}\right)\left(\dfrac{1.21 \times 10^{-10}}{2}\right)^2} \sim 10^{12} rad/s$$